A THEOREM ON VALUATION RINGS AND ITS APPLICATIONS

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To Professor K. NOSHIRO on his 60th birthday

In the present paper, we first prove the following

THEOREM 1. *Let K be a field, x a transcendental element over K, and V^* a valuation ring of $K(x)$. Set $V = V^* \cap K$. Denote by $\mathfrak{p}^*$ and $\mathfrak{p}$ the maximal ideals of V^* and V respectively. If* (i) *$V^*/\mathfrak{p}^*$ is not algebraic over $V/\mathfrak{p}$ and* (ii) *the value group of V^* is isomorphic to $\mathbf{Z}^n$ ($\mathbf{Z}$ = the module of rational integers), i.e., V^* is of rank n and discrete in the generalized sense,[2] then $V^*/\mathfrak{p}^*$ is a simple transcendental extension of a finite algebraic extension of $V/\mathfrak{p}$.*

Then we show some applications of this theorem to the theory of fields. At the end of this paper, we shall discuss the theorem above without assuming (ii).

Besides usual terminology on rings and fields, we make the following definitions: (1) a field L is said to be *ruled* over its subfield K if L is a simple transcendental extension of its subfield containing K, (2) a field L is said to be *anti-rational* over its subfield K if no finite algebraic extension of L is ruled over K and (3) a field L is said to be *quasi-rational* over its subfield K if every subfield of L which contains K properly is not anti-rational over K.

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1. Proof of Theorem 1

We use induction argument on n. If $n = 0$, then the assertion is obvious.

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2) Under the presence of the condition (i), this (ii) is equivalent to that the value group of V is isomorphic to $\mathbf{Z}^n$, as is easily seen from the fact that $V(y) = V[y]_{\mathfrak{p}V[y]}$ is a valuation ring dominated by V^* when $y \in V^*$ is transcendental modulo $\mathfrak{p}^*$ over $V/\mathfrak{p}$.