LIFTING PROJECTIVES

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In memory of TADASI NAKAYAMA

1. Introduction and statement of result

Let R be a ring with radical $\mathfrak{N}$ (all rings have a unit element, all modules are unital). Often, one wishes to lift modules modulo $\mathfrak{N}$, that is, to a given, say, left $R/\mathfrak{N}$-module U find a left R-module E with the property that $E/\mathfrak{N}E \simeq U$. This is of course not always possible. Here I prove, roughly, that if a finitely generated projective U can be lifted at all, it can be lifted to a projective. Or rather, if U can be lifted to an E satisfying a certain mild condition, then E is projective (Lemma).

It is convenient to introduce the notion of "cover". In any category, an epimorphism $f : A \to B$ is called a cover if any morphism $g : X \to A$ such that fg is an epimorphism, must needs be an epimorphism. Sloppily, we also say that A is a cover of B. In the category of R-modules, Nakayama's Lemma asserts that f is a cover if A is finitely generated and $\ker f \subset \mathfrak{N}A$. Repeated application of this simple remark will prove the result, which I dedicate to the memory of T. Nakayama.

LEMMA. *Let R be a left noetherian ring, $\mathfrak{A}$ a two-sided ideal contained in its radical. Let U be a finitely generated projective $R/\mathfrak{A}$-module. Suppose the left R-module E is an R-cover of U and that* $\mathrm{Tor}_1^R(R/\mathfrak{A}, E) = 0$. *Then E, uniquely determined up to isomorphism, is finitely generated projective. Moreover, $E/\mathfrak{A}E \simeq U$.*

This fact is useful in the theory of homological dimension. For commutative rings, it is easily derived from the "critère de platitude" [4, Ch. III, Th. 1, p. 98], bearing in mind that finitely presented flat modules are projective. Even here, however, the approach using covers is more direct. A variant of the lemma was proved in [8, Lemma 1.13, p. 6] with a different application in view. Since theses are seldom produced in order to be read, it seems worth

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