A THEOREM OF HARRISON, KUMMER THEORY, AND GALOIS ALGEBRAS

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To the memory of TADASI NAKAYAMA

1. Introduction. Let R be a field and S a separable algebraic closure of R with galois group $\mathfrak{G}_R$. In [8] Harrison succeeded in describing $\mathfrak{G}_R/\mathfrak{G}'_R$ in terms of R only. More precisely, he constructed a certain complex $\mathfrak{H}(R, Q/Z)$ and proved $\mathrm{Hom}_c(\mathfrak{G}_R, Q/Z) \cong H^2(R, Q/Z)$, where Hom_c denotes continuous homomorphisms and H^2 stands for the second cohomology group of the complex $\mathfrak{H}$. In this paper, which is mainly expository in nature, we reexamine Harrison's proof and show how [8] connects with Kummer theory and the theory of galois algebras [16]. We emphasize that most of the ideas on which this paper is based originate in [8].

In 2. the complex $\mathfrak{H}(R, J)$ is introduced for any commutative ring R and abelian group J. If J is a finite abelian group of exponent e and R a field of characteristic prime to e, we show that $\mathfrak{H}(R, J)$ is isomorphic to a complex arising in ordinary group cohomology. This enables us to compute $H^n(R, J)$ in case R is a field of characteristic prime to e and containing the e^{th}-roots of unity. In 3. we consider a field R and a galois extension field S with galois group $\mathfrak{G}$. We give two proofs, one using spectral sequences, of the existence of an exact sequence

$$0 \to \mathrm{Hom}_c(\mathfrak{G}, J) \to H^2(R, J) \to H^2(S, J).$$

From this we can already deduce Harrison's result in case R has characteristic 0 and also obtain the main exact sequence of Kummer theory. Section 4 is devoted to a new exposition of the foundation of the theory of galois algebras [16] over a commutative ring. It turns out that the dual concept, that of galois coalgebra, is much more amenable to treatment and we accordingly

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