RADICAL MODULES OVER A DEDEKIND DOMAIN

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To the memory of TADASI NAKAYAMA

A radical of a field K is a non zero element of a given algebraic closure some positive power of which lies in K. The group $R(K)$ of radicals reflects properties of the field K and is in turn easily determined as an extension of the multiplicative group K^* of non zero elements of K. The elements of the quotient group $R(K)/K^*$ are then conveniently identified with certain subspaces of the algebraic closure, the radical spaces of K (cf. § 1). What we are here concerned with is the corresponding arithmetic situation, in which we start with a Dedekind domain $\mathfrak{o}$ with quotient field K. The role of the radicals is taken over by the radical modules. These form a group $\mathfrak{R}(\mathfrak{o})$ which contains the group of fractional ideals of $\mathfrak{o}$ (cf. § 4).

The ideal theory of $\mathfrak{o}$ is equivalent with its valuation theory. Although the same is no longer strictly true, when translated into the new context, there is still a close connection between the "valuations" of $R(K)$ and the theory of radical modules (cf. § 3, 4). There is also the new feature, to which there is no analogue in the valuation theory of $\mathfrak{o}$, that each discrete valuation gives rise to a character of the radical spaces, i.e. of $R(K)/K^*$. These characters can in a natural manner be lifted back to real valued functions. On the basis of the theory of integral radical modules—the analogue to the integral ideals—one then obtains the conductor of a radical space (§ 6), which plays an important role in the ramification theory.

The group $\mathfrak{R}(\mathfrak{o})$ differs from $R(K)$ in presenting a genuine divisibility problem. We shall show, in § 5, that a radical module is divisible by n if and only if its image under a canonical map onto the ideal class group of $\mathfrak{o}$ is divisible by n. Looking at it the other way round, one obtains in terms of $\mathfrak{R}(\mathfrak{o})$ an essentially local, necessary and sufficient condition for the class of a fractional ideal of $\mathfrak{o}$ to be an n-th power. As an application of this criterion we shall then give a

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