SIMPLE ALGEBRAS OVER A COMMUTATIVE RING

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In memory of Professor TADASI NAKAYAMA

In a previous paper [4], we studied a class of algebras over a commutative ring R which we called semisimple algebras. Here we shall study simple algebras.

In [4], we defined simple algebras over a Prüfer domain R as those semisimple algebras whose rational hulls are simple. A simple algebra A in this sense is directly indecomposable (as an algebra) and any (one-sided) ideal is A-faithful. In agreement with this, we defined simple algebras over a general commutative ring as semisimple algebras A admitting A-faithful and (A, R)-irreducible modules M (definition below), in the Symposium 1964 in Sapporo. But, in studying such simple algebras, we need very often that the natural monomorphism $A \to \mathrm{Hom}_R(M, M)$ admits an R-splitting. In this paper, we include this property in the definition of simplicity.

1. Let A be an algebra with an identity over a commutative ring R (with identity). A left A-module M is called (A, R)*-irreducible* if M has no non-trivial A-submodule which is an R-direct summand. If R is semisimple, (A, R)-irreducibility is identical with A-irreducibility. In [4], we introduced the notion of a semisimple algebra over R. If A is left semisimple over R, then (A, R)-irreducibility coincides with A-indecomposability. It follows at once the following proposition.

PROPOSITION 1. *Let A be an R-finite semisimple algebra over a Noetherian ring R. Then, a finitely generated left A-module is a direct sum of a finite number of (A, R)-irreducible modules. In particular, A itself is a direct sum of a finite number of (A, R)-irreducible left ideals.*

2. A left A-module M is called *completely faithful* if its trace ideal [1] coincides with A. This means that there exist a finite number of elements

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