ON REGULAR SEQUENCES

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In memory of TADASI NAKAYAMA

1. The concept of *regular sequence* of elements of a ring A (first introduced by Serre under the name of A-sequence [2]), has far-reaching uses in the theory of local rings and in algebraic geometry. It seems, however, that it loses much of its importance when A is not a noetherian ring, and in that case, it probably should be superseded by the concept of *quasi-regular sequence* [1].

One of the convenient properties of a regular sequence $t_1, \ldots, t_n$ in a noetherian local ring A, where the t_i's belong to the maximal ideal of A, is that the sequence remains regular after an arbitrary *permutation* of its terms. This is due to the fact that in such a case, the notions of regular sequence and of quasi-regular sequence coincide [1, 15.1.10], and the notion of quasi-regular sequence is independent of the order of the elements of the sequence.

I will give below an example of a non noetherian local ring A and of two elements t_1, t_2 of the maximal ideal of A, such that the sequence (t_1, t_2) is regular, whilst the sequence (t_2, t_1) is not. Such unpleasant phenomena greatly reduce the usefulness of the notion of regular sequence.

2. To construct our example, we start with the ring B of all germs of indefinitely differentiable functions of a real variable x in the neighborhood of 0. It is well known that B is a local ring, whose maximal ideal $\mathfrak{n}$ is generated by the germ i of the identity mapping $x \to x$; the intersection of all powers $\mathfrak{n}^k$ $(k = 1, 2, \ldots)$ is the ideal $\mathfrak{r} \neq 0$ consisting of all germs of functions whose derivatives all vanish at $x = 0$. Observe that the complement of $\mathfrak{r}$ in B consists of regular elements of B (i.e. elements which are not zero-divisors).

Now consider the ring of polynomials $B[T]$ in one indeterminate, and let C be the quotient of $B[T]$ by the ideal $\mathfrak{r}TB[T]$, consisting of all polynomials $r_1T + \cdots + r_mT^m$ having their coefficients in $\mathfrak{r}$. We prove that in C, the

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