A NOTE ON IDEAL CLASS GROUPS

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To the memory of TADASI NAKAYAMA

In the first part of the present paper, we shall make some simple observations on the ideal class groups of algebraic number fields, following the group-theoretical method of Tschebotarew[1]. The applications on cyclotomic fields (Theorems 5, 6) may be of some interest. In the last section, we shall give a proof to a theorem of Kummer on the ideal class group of a cyclotomic field.

1. For any prime numbers p and q, let

$$\begin{aligned} d(q,p) &= 2, && \text{for } p = q, \\ &= \text{the order of } p \bmod q, && \text{for } p \neq q. \end{aligned}$$

For any integer $n \geq 1$, we then define

$$d(n,p) = \text{the minimum of } d(q,p) \text{ for all prime factors } q \text{ of } n.$$

THEOREM 1. *Let G be a finite group of order n. Let M be a G-module over the prime field P with p elements, and let d be the dimension of M over P. Suppose that the action of G on M is non-trivial. Then*

$$d \geqq d(n,p).$$

Proof. Let σ be an element with minimal order in G such that the action of σ on M is non-trivial. Let q be a prime dividing the order of σ. Put $H = G_1/G_2$, where G_1 and G_2 denote the subgroups of G generated by σ and σ^q respectively. Then M is also an H-module over P, and the action of H on M is non-trivial. If $q = p$, we see immediately that $d \geqq 2 = d(p,p)$. Suppose that $q \neq p$. Then M is completely reducible, and it has an irreducible submodule on

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1) N. Tschebotarew, Zur Gruppentheorie des Klassenkörpers, J reine u. angew. Math., **161** (1929), pp. 179–183.