HEREDITARY LOCAL RINGS

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To the memory of TADASI NAKAYAMA

1. Introduction

Many questions about free ideal rings (= firs, cf. [5] and § 2 below) which at present seem difficult become much easier when one restricts attention to local rings. One is then dealing with hereditary local rings, and any such ring is in fact a fir (§ 2). Our object thus is to describe hereditary local rings. The results on firs in [5] show that such a ring must be a unique factorization domain; in § 3 we prove that it must also be rigid (cf. the definition in [3] and § 3 below). More precisely, for a semifir[1] R with prime factorization rigidity is necessary and sufficient for R to be a local ring.

§ 4 gives an example of a right fir (in fact a principal right ideal domain) with prime factorization, which is not left hereditary and hence is not a left fir. Since the example is of a local ring, this provides an example of a rigid unique factorization domain which is a semifir but not a fir.

The final section concerns the centre of a hereditary local ring. If this is not a field, then both the ring and its centre are discrete valuation rings. This improves a result of Northcott [8] who showed that the centre, if not a field, must be a 1-dimensional regular local ring. The actual result proved in § 5 is rather more general (apart from the stronger conclusion) in that the hypothesis is weaker: we do not require the existence of a central non-unit ($\neq 0$) but merely a 'large' non-unit, and in an integral domain every central non-unit $\neq 0$ is large.

2. Hereditary and semihereditary local rings

Throughout, all rings are associative with 1, and all modules are unital. We recall that a ring R is said to be *p-trivial* (= projective-trivial) if there is

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[1] Semifirs, called 'local firs' in [5] have been renamed here (by analogy with 'semi-hereditary') to avoid confusion with local rings.