RINGS WITH ASCENDING CONDITION ON ANNIHILATORS

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Quasi-frobenius ($=QF$) rings have many interesting characterizations. One such, due to Ikeda [17] is that these rings are right (left) artinian and right (left) self-injective. Thus, if R is QF, then R is right (left) noetherian, so each direct sum of injective right R-modules is injective: in particular, each free, hence, each projective, R-module is injective. One object of this paper is to report that this property characterizes QF-rings:

(A) THEOREM. *A ring R is QF if and only if each projective right R-module is injective.*

The symmetrical properties of QF-rings (§2) show that "right" can be replaced by "left" in this statement. The "dual" theorem obtained by the substitutions "projective" $\leftrightarrow$ "injective" is the subject of another paper [7].

The condition that every free module is injective leads naturally to the concept of Σ-injectivity: an injective module is ***Σ-injective*** in case an infinite direct sum of copies is injective. A Σ-injective module M_R with endomorphism ring Λ is characterized by the descending chain condition (d.c.c.) on the lattice of Λ-submodules which are annihilators of subsets of R (Prop. 3.3). If $\hat{R}$ denotes the injective hull of R_R, and if $M=\hat{R}$, this condition implies the ascending chain condition (a.c.c.) on annihilator right ideals ($=$ right annulets) of R, and, in case $M=\hat{R}=R$, this condition is equivalent to the a.c.c. on right annulets (Corollary 3.4 and Theorem 3.5). Thus, the proof of (A) leads to the more general study of the rings of the title, and to the following intrinsic characterization: R satisfies the a.c.c. on right annulets if and only if to each right ideal I there corresponds a finitely generated subideal I_1 having the same left annihilator as I (Prop. 3.1).

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