INVOLUTORY AUTOMORPHISMS OF GROUPS OF ODD ORDER AND THEIR FIXED POINT GROUPS

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In Memoriam Tadasi Nakayama

1. **Introduction.** Let G be a finite group of odd order with an automorphism θ of order 2. (We use without further reference the fact, established by W. Feit and J. G. Thompson, that all groups of odd order are soluble.) Let G_θ denote the subgroup of G formed by the elements fixed under θ. It is an elementary result that if $G_\theta=1$ then G is abelian. But if we merely postulate that G_θ be cyclic, the structure of G may be considerably more complicated—indeed G may have arbitrarily large soluble length. E.g. let p be an odd prime and let t denote the largest odd divisor of $p-1$. Let G be the group formed by the matrices $A=\begin{pmatrix} u+pa & pb \\ pc & v+pd \end{pmatrix}$ of determinant 1, where a, b, c, d, u, v lie in the ring of residue classes (mod p^{k+1}) and $uv\equiv u^t\equiv 1$ (mod p). Let θ be the contragradient automorphism $A\to(A^{-1})^T$. Then G_θ is cyclic of order p^k. G itself has order $p^{3k}t$ and soluble length m or $m+1$, where m is the least integer such that $2^m\geq k+1$. The Fitting subgroup of G is a p-group of order p^{3k}, exponent p^k, and class k.

The theorem proved in this note deals with the case where G_θ is nilpotent. It belongs to the same circle of ideas as the recent results of J. G. Thompson [5], though it is much more special. Let $F(H)$ denote the Fitting subgroup of a group H and $1=F_0(H)\leq F_1(H)\leq\cdots$ the ascending Fitting series of H, defined inductively by $F_{i+1}(H)/F_i(H)=F(G/F_i(H))$.

Theorem Let G be a group of odd order and θ an automorphism of G of order 2 such that G_θ is nilpotent. Suppose either (i) $G=F_2(G)$ or (ii) the Sylow subgroups of G_θ are regular. Then $G/F(G)$ is contained in the variety $\mathbf{V}$ generated by G_θ together with the cyclic subgroups of $G/F(G)$.

The following particular case may serve as an illustration of the Theorem:

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