AN ACTION OF THE SYMPLECTIC MODULAR GROUP

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To the memory of TADASI NAKAYAMA

1. Let V be a free $\mathbf{Z}$-module of rank $2n$. Let $G = \mathbf{Sp}(2n, \mathbf{Z})$ be the symplectic modular group and let Φ be the non-singular alternating bilinear form on V left invariant by G. Let $p \in \mathbf{Z}$ be a prime and let X be the set of all endomorphisms ξ of V such that

$$\Phi(\xi x, \xi y) = p\Phi(x, y)$$

for all $x, y \in V$. In the theory of transformation of theta functions [3] one encounters the natural action of G on X by left multiplication. The number of G orbits is known to be finite and the point of this note is a proof of the following

THEOREM. *The number of orbits of X under G is* $\prod_{i=1}^{n}(1+p^i)$

The case $n = 2$ is due to Hermite [2] and the case $n = 3$ to Weber [4] who compute explicit sets of representatives for the orbits in these cases. The idea in the present argument is to reduce the problem to a question about the finite symplectic group $\mathbf{Sp}(2n, \mathbf{F}_p)$. In the new situation Witt's theorem is available for counting purposes. The number $\prod_{i=1}^{n}(1+p^i)$ is the number of maximal totally isotropic subspaces of a $2n$ dimensional symplectic space over $\mathbf{F}_p$.

2. Let V be a free $\mathbf{Z}$-module of rank $2n$. Let Φ: $V \times V \to \mathbf{Z}$ be a non-singular alternating bilinear form on V. We assume that V has a basis $v_1, \ldots, v_{2n}$ such that the matrix of $\Phi(v_i, v_j)$ is

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the identity matrix of degree n. We call $v_1, \ldots, v_{2n}$ a symplectic basis for V. The symplectic modular group $\mathbf{Sp}(2n, \mathbf{Z})$ consists of all automor-

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