ON SOME INFINITE DIMENSIONAL REPRESENTATIONS OF SEMI-SIMPLE LIE ALGEBRAS

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1. Introduction

Let $\mathfrak{g}$ be a semi-simple Lie algebra over an algebraically closed field K of characteristic 0. For finite dimensional representations of $\mathfrak{g}$, the following important results are known;

1) $H^1(\mathfrak{g}, V)=0$ for any finite dimensional $\mathfrak{g}$ space V. This is equivalent to the complete reducibility of all the finite dimensional representations.

2) Determination of all irreducible representations in connection with their highest weights.

3) Weyl's formula for the character of irreducible representations [9].

4) Kostant's formula for the multiplicity of weights of irreducible representations [6].

5) The law of the decomposition of the tensor product of two irreducible representations [1].

Harish-Chandra [3] studied the infinite dimensional $\mathfrak{g}$-spaces with dominant vectors, and established 2) and 3) for such spaces. The lacking of the complete reducibility 1) necessitates the study of $\mathrm{Ext}^1_{\mathfrak{g}}(U, V)$. A. Hattori determined the structure of $H^1(\mathfrak{g}, V)=\mathrm{Ext}^1_{\mathfrak{g}}(K, V)$ for an irreducible $\mathfrak{g}$-space V with a dominant vector [4]. To study the general case we need more. If U is finite dimensional, we have

$$\mathrm{Ext}^1_{\mathfrak{g}}(U, V)=H^1(\mathfrak{g}, \mathrm{Hom}_K(U, V))=H^1(\mathfrak{g}, U^*\otimes V)$$

where U^* is the contragredient representation of U, so that we are led to the study of $U^*\otimes V$, a special case of 5). Theorem 1 of § 2 concerns with the structure of the tensor product. It follows from this theorem together with a generalization of Hattori's result (Theorem 2) that $\mathrm{Ext}^1_{\mathfrak{g}}(U, V)=0$ for certain U and V (Theorem 3 of § 3). In § 4 we obtain a formula for the multiplicity of weights in case $\mathfrak{g}=\mathfrak{sl}(3, K)$.

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