RANK ELEMENT OF A PROJECTIVE MODULE

AKIRA HATTORI

In § 1 of this note we first define the trace of an endomorphism of a projective module P over a non-commutative ring A. Then we call the trace of the identity the rank element $r(P)$ of P, which we shall illustrate by several examples. For a projective module P over the groupalgebra of a finite group G, the rank element of P is essentially the character of G in P. In § 2 we prove that under certain assumption two projective modules P_1 and P_2 over an algebra over a complete local ring $\mathfrak{o}$ are isomorphic if and only if their rank elements are identical. This is a type of proposition asserting that two representations are equivalent if and only if their characters are identical, and in fact, when A is the groupalgebra, the above theorem may be considered as another formulation of Swan's local theorem [9]*).

§ 1. Trace and rank element

1.1. Let A be a ring with an identity, and $[A, A]$ its commutator, i.e. the set of finite sums of elements of the form $ab - ba$ $(a, b \in A)$. We denote the abelian group $A/[A, A]$ by A^a, and the natural epimorphism $A \to A^a$ by ε. ε is symmetric: $\varepsilon(ab) = \varepsilon(ba)$.

For a left A-module M, we have the right A-module $\mathrm{Hom}_A(M, A)$, called the dual module of M and denoted by M^* in this paper. It is easy to check that the pairing $(\xi, x) \to \varepsilon(\xi(x))$ $(\xi \in M^*, x \in M)$ induces a well-defined homomorphism $M^* \otimes_A M \to A^a$, which we shall denote by π.

Let P be a finitely generated projective left A-module. This means that the mapping $\theta : P^* \otimes_A P \to \mathrm{Hom}_A(P, P)$ defined by $\theta(\xi \otimes x)(y) = \xi(y)x$ $(x, y \in P, \xi \in P^*)$ is an isomorphism. Now, we shall define the *trace* $\mathrm{Tr}_A(f)$ (or simply $\mathrm{Tr}(f)$) of an endomorphism $f \in \mathrm{Hom}_A(P, P)$ by $\mathrm{Tr}(f) = \pi(\theta^{-1}(f))$, which is a

Received January 25, 1964.

*) The present work was done in 1960–1961, and briefly announced in [5]. Our main objective was to study Swan's results without use of Grothendieck rings. In the meantime, there appeared the works of Giorgiutti [4], Rim, and Bass [2], which gave the same problem nice answers together with a generalization.