INVARIANTS OF FINITE REFLECTION GROUPS

LOUIS SOLOMON

To RICHARD BRAUER on his 60th birthday

1. Let K be a field of characteristic zero. Let V be an n-dimensional vector space over K and let S be the graded ring of polynomial functions on V. If G is a group of linear transformations of V, then G acts naturally as a group of automorphisms of S if we define

$$(\gamma s)(v) = s(\gamma^{-1} v) \qquad \gamma \in G,\ s \in S,\ v \in V$$

The elements of S invariant under all $\gamma \in G$ constitute a homogeneous subring $I(S)$ of S called the ring of polynomial invariants of G.

A linear transformation of V is a reflection if it has finite order and leaves fixed an $n-1$ dimensional subspace, its reflecting hyperplane. If G has finite order and is generated by reflections we call it a finite reflection group. For such groups we know from work of Chevalley [2] and Coxeter [3] that the ring $I(S)$ is a polynomial ring generated by n algebraically independent forms $f_1, \ldots, f_n$. In fact, Shephard and Todd [4] have shown that this property of the ring of polynomial invariants characterizes the finite groups generated by reflections. It has been known for a long time, at least for the real orthogonal groups, that the degrees $m_1 + 1, \ldots, m_n + 1$ of the forms $f_1, \ldots, f_n$ satisfy the product formula $(m_1 + 1) \cdots (m_n + 1) = g$, where g is the order of G, and that the sum $m_1 + \cdots + m_n$ is equal to the number of reflections in the group. More recently, Shephard and Todd [4] discovered and verified the general formula

$$(1) \qquad (1 + m_1 t) \cdots (1 + m_n t) = g_0 + g_1 t + \cdots + g_n t^n$$

where g_r is the number of elements of G that fix some $n-r$ dimensional subspace of V but fix no subspace of higher dimension. If G is a crystallographic group then the Poincaré polynomial of the corresponding Lie group is known to be $(1 + t^{2m_1+1}) \cdots (1 + t^{2m_n+1})$ so that the formula yields a method for com-

Received May 21, 1962.