ON A CLASS OF CONFORMAL METRICS

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Last year when I was preparing for course lectures the work of Ahlfors [1] which establishes that the Bloch constant is at least as large as $\sqrt{3}/4$, it appeared to me that the resources of the theory of metrics of negative curvature offered rich possibilities from a function-theoretic point of view. The parallelism between certain properties of subharmonic functions and those of the metrics introduced by Ahlfors [1] is so striking that we are led to ask whether one can introduce a class of metrics including the metrics of Ahlfors for which not only does a Schwarz-Pick-Ahlfors lemma hold, but also requirements of differentiability disappear, as in the modern theory of subharmonic functions. We shall define such a class. To it part of the apparatus of the theory of subharmonic functions, including the use of Perron families, may be transplanted. Among the results that we obtain is the conclusion that the inequality of the Schwarz-Pick-Ahlfors lemma is *strong throughout* for an admitted metric distinct from the hyperbolic metric [§ 7]. This theorem will permit us to show that the Bloch constant is *actually greater than* $\sqrt{3}/4$ [§ 36].

A metric of special interest from the point of view of the theory of conformal maps of Riemann surfaces is one having constant curvature -4 save for a discrete set at each point of which it vanishes. Metrics induced by a conformal map from a hyperbolic metric are of this type. We shall see that the distribution of the zeros of the derivative of a non-constant bounded analytic function with domain $\Delta = \{|z| < 1\}$ may be characterized in terms of such a metric [§ 29]. In this connection, the following result deserves mention: The distribution of points in Δ at which a Lindelöfian map with domain Δ is ramified, multiplicities being taken into account, is no more general than the distribution of the ramification points of a non-constant bounded analytic function with domain Δ[§ 30].

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