THETA FUNCTIONS AND ABELIAN VARIETIES OVER VALUATION FIELDS OF RANK ONE I.

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Introduction. We shall denote by $\mathfrak{M}$ the Z-module of integral vectors of dimension r, by T a symmetric complex matrix with positive definite imaginary part and by $\mathfrak{z}$ the variable vector. If we put $q(\mathfrak{m}, \mathfrak{n}) = e^{\pi\sqrt{-1}\,{}^t\mathfrak{m}T\mathfrak{n}}$ and $u(\mathfrak{m}) = e^{2\pi\sqrt{-1}\,{}^t\mathfrak{m}\mathfrak{z}}(\mathfrak{m}, \mathfrak{n} \in \mathfrak{M})$ the fundamental theta function $\sum_{\mathfrak{m}\in\mathfrak{M}} e^{\pi\sqrt{-1}\,{}^t\mathfrak{m}T\mathfrak{m}+2\pi\sqrt{-1}\,{}^t\mathfrak{m}\mathfrak{z}}$ is expressed in the form: $\vartheta(q|u) = \sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}, \mathfrak{m})u(\mathfrak{m})$ as a series in q and u. Other theta functions in the classical theory are derived from the fundamental theta function by translating the origin and making sums and products, so these theta functions are also expressed in the form: $\sum_{\mathfrak{m}\in\mathfrak{M}} c_{\mathfrak{m}}q(\mathfrak{m}, n^{-1}\mathfrak{m})u(\mathfrak{m})$ as series of q and u. Moreover the coefficients in the relations of theta functions are also expressed in the form: $\sum_{\mathfrak{m}\in\mathfrak{M}} d_{\mathfrak{m}}q(\mathfrak{m}, n^{-1}\mathfrak{m})$ as series in q.

All the parts of theory of theta functions are formal except only one point: The products of theta functions are also theta functions. This property of the products of theta functions comes out as a result from the positive definiteness of the imaginary part of T. The positive definiteness of the imaginary part of T is equivalent to the condition: $|q(\mathfrak{m}, \mathfrak{m})|<1$ for $\mathfrak{m} \neq 0$.

This situation suggests the possibility of replacement of the field of complex numbers with a field complete with respect to a valuation of rank one.

§ 1. Summary and notations.

1. 1. We mean by a valuation v of rank one of a field Ω a mapping v of Ω into the additive group of real numbers satisfying $v(\xi\eta) = v(\xi) + v(\eta)$ and $v(\xi+\eta) \geq \min\{v(\xi), v(\eta)\}(\xi, \eta \neq 0$ in $\Omega)$.

We fix, once for all in the following, an algebraically closed field Ω complete with respect to a (non-trivial) valuation v of rank one. We also choose a field K containing Ω such that 1° K is algebraically closed and 2° K contains infinite many elements $x_1, x_2, \ldots$ such that there exists no relation $\sum_{i_1,\ldots,i_r\in n^{-1}Z} a_{i_1\cdots i_r}x_1^{i_1}$

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