ON UNIT GROUPS OF ABSOLUTE ABELIAN NUMBER FIELDS OF DEGREE pq

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In this note, we denote by Q the rational number field, by $\mathbf{E}_\Omega$ the whole unit group of an arbitrary number field Ω of finite degree, and by r_Ω the rank of $\mathbf{E}_\Omega^*$, where generally $\mathbf{G}^*$ for an arbitrary abelian group $\mathbf{G}$ means a maximal torsion-free subgroup of $\mathbf{G}$. $(N_{K/\Omega}\mathbf{E}_K)^*$ is shortly denoted by $N_{K/\Omega}^*\mathbf{E}_K$ and $(\mathbf{G}_1 : \mathbf{G}_2)$ is, as usual, the index of a subgroup $\mathbf{G}_2$ in $\mathbf{G}_1$.

We first prove the following lemma.

LEMMA. *Let* $\mathbf{F}$ *be a free abelian group of finite rank* n, *and* $\mathbf{G}$ *be a subgroup of* $\mathbf{F}$ *such that for a rational prime number* l, $\mathbf{G}$ *contains the group* $\mathbf{F}^l$ *consisting of all the* l*-th powers* α^l *of* α *in* $\mathbf{F}$. *Then, for an arbitrarily given basis* $(\varepsilon_1, \ldots, \varepsilon_n)$ *of* $\mathbf{F}$, $\mathbf{G}$ *has the basis* $(\omega_1, \ldots, \omega_n)$ *of the following form:*

$$\omega_i = \begin{cases} \varepsilon_{\pi_i}^l \cdots\cdots\cdots\cdots & i = 1, \ldots, s, \ (s \geqq 0) \\ \varepsilon_{\pi_i} \prod_{j=1}^{s} \varepsilon_{\pi_j}^{a_{ij}} \cdots\cdots\cdots & i = s+1, \ldots, n, \end{cases}$$

where a_{ij} *are rational integers with* $0 \leqq a_{ij} < l$ *and* $(\pi_1, \ldots, \pi_n)$ *is a suitable permutation of* $(1, \ldots, n)$.

Proof. By the elementary divisor theory, there exist a basis $(f_1, \ldots, f_n)$ of $\mathbf{F}$ and a basis $(g_1, \ldots, g_n)$ of $\mathbf{G}$ such that we may write $(g_1, \ldots, g_n) = (f_1, \ldots, f_n)L$, where L is a $n \times n$ diagonal matrix with diagonal elements e_{i+1}/e_i $(i = 1, \ldots, n-1)$. By the assumption, however, all the l-th powers of the elements in $\mathbf{F}$ are contained in $\mathbf{G}$, so we have $e_1 = \cdots = e_s = l$, $e_{s+1} = \cdots = e_n = 1$ for some integer s $(0 \leqq s \leqq n)$. We express this basis $(f_1, \ldots, f_n)$ of $\mathbf{F}$ by using the basis $(\varepsilon_1, \ldots, \varepsilon_n)$ of $\mathbf{F}$:

$$(f_1, \ldots, f_n) = (\varepsilon_1, \ldots, \varepsilon_n)U,$$

where U is an unimodular matrix of degree n.

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