A REMARK ON RELATIVE HOMOLOGY AND COHOMOLOGY GROUPS OF A GROUP

To Zyoiti Suetuna on his 60th Birthday

TADASI NAKAYAMA

Let G be a group and H a subgroup of G. With a left G-module M, relative cohomology groups $H^n(G, H, M)$ of G on M, relative to H, have been defined by Adamson [1] and may be expressed as $\mathrm{Ext}^n_{(G, H)}(Z, M)$ in the notation of relative homological algebra of Hochschild [2], where Z denotes the G-module of rational integers (acted by G trivially). Regarding M as a right G-module, $\mathrm{Tor}_n^{(G, H)}(M, Z)$ are similarly relative homology groups $H_n(G, H, M)$. In case H is of finite index in G, Hochschild [2] defines further negative-dimensional relative homology and cohomology groups. He then remarks that these complete relative homology and cohomology structures are separate (contrary to the absolute case $H=1$). Indeed he exhibits an example of G, H, M (with H even normal in G) such that $H^n(G, H, M)=0$ for every $n=0, \pm 1, \pm 2, \ldots$ and $H_n(G, H, M)$ is a group of order 2 for every $n=0, \pm 1, \pm 2, \ldots$. This, however, does not exclude the possibility that negative-dimensional relative homology groups $H_{-n}(G, H, M)$ are in close relationship with positive-dimensional relative cohomology groups on some G-module N other than M. In fact, in case H is a normal subgroup of G, we have $H_{-n}(G, H, M) \approx H_{-n}(G/H, M_H) \approx H^{n-1}(G/H, M_H)$ (where M_H denotes as usual the residue-module of M with respect to the submodule generated by the elements of form $u-hu$ ($u \in M$, $h \in H$) and this is isomorphic to $H^{n-1}(G/H, N^H) \approx H^{n-1}(G, H, N)$ if M_H is G-isomorphic to N^H (where N^H is the submodule of N consisting of all elements of N left invariant by H); this holds not only for $n>0$ but for all $n=0, \pm 1, \pm 2, \ldots$. Now we want to show that a similar phenomenon prevails also in case of a non-normal subgroup H.

Thus, let H be a subgroup of finite index in a group G and K_0 be the largest normal subgroup of G contained in H, i.e. the intersection of all conjugates of H in G. For G-modules M and N, we consider the following condition:

Received September 7, 1959.