INJECTIVE MODULES OVER PRUFER RINGS

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The purpose of this paper is to find out what can be learned about valuation rings, and more generally Prufer rings, from a study of their injective modules. The concept of an almost maximal valuation ring can be reformulated as a valuation ring such that the images of its quotient field are injective. The integral domains with this latter property are found to be the Prufer rings with a (possibly) weakened form of linear precompactness for their quotient fields. The Prufer rings with linearly compact quotient fields are found to be exactly the maximal valuation rings, and may be characterized as those integral domains R with quotient field Q such that the images of Q are injective and $\mathrm{Hom}_R(Q/R, Q/R) \cong R$; or, alternatively, as those integral domains for which a torsion-free submodule S of rank one of a module B is a direct summand whenever B/S is also torsion-free. We are able to rederive many of the results of [2] and [4] by homological methods. Finally, among Noetherian integral domains we characterize the Dedekind rings as those for which every finitely generated torsion module is a direct sum of cyclic modules.

Notation and Definitions. Any ring considered will be commutative with an identity element which acts as the identity operator on any module over the ring. If A is any module, we will denote by $E(A)$ the injective envelope of A [6]. A module A is said to be *indecomposable*, if it has no proper direct summands. An ideal I of a ring is said to be *irreducible*, if it is not an intersection of two properly larger ideals. A module over an integral domain will be said to be *divisible*, if multiplication by any non-zero element of the ring is an epimorphism of the module onto itself; and a module will be said to be *torsion-free*, if any such multiplication is a monomorphism of the module into itself.

A *Prufer ring* is an integral domain in which every finitely generated ideal is invertible. A *valuation ring* is an integral domain in which every two ele-

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