ON THE DIMENSION OF MODULES AND ALGEBRAS, IV

DIMENSION OF RESIDUE RINGS OF HEREDITARY RINGS

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A ring (with unit element) Λ is called semi-primary[1] if it contains a nilpotent two-sided ideal N such that the residue ring $\Gamma = \Lambda/N$ is semi-simple (i.e. $\text{l.gl.dim}\,\Gamma = \text{r.gl.dim}\,\Gamma = 0$). N is then the (Jacobson) radical of Λ. Auslander [1] has shown that if Λ is semi-primary then

$$\text{l.gl.dim}\,\Lambda = \text{r.gl.dim}\,\Lambda = \text{l.dim}_\Lambda\,\Gamma = 1 + \text{l.dim}_\Lambda\,N.$$

The common value is denoted by $\text{gl.dim}\,\Lambda$. On the other hand, for any ring Λ the following conditions are equivalent: (a) $\text{l.gl.dim}\,\Lambda \leqq 1$, (b) each left ideal in Λ is projective, (c) every submodule of a projective left Λ-module is projective. Rings satisfying conditions (a)-(c) are called *hereditary*. For integral domains the notions of "hereditary ring" and "Dedekind ring" coincide.

If $\mathfrak{a}$ is a two-sided ideal in Λ, there is in general very little relation between $\text{l.gl.dim}\,\Lambda$ and $\text{l.gl.dim}\,(\Lambda/\mathfrak{a})$; it was however proved, substantially, in Eilenberg-Ikeda-Nakayama [4] that if Λ is semiprimary and $\mathfrak{a}$ is contained in the radical N then

$$\text{gl.dim}\,\Lambda \leqq \text{gl.dim}\,(\Lambda/\mathfrak{a}) + \text{l.dim}_\Lambda\,(\Lambda/\mathfrak{a}).$$

Now, we show in §1, of the present note, that if Λ is hereditary and the sequence $\mathfrak{a}^i$ ($i = 1, 2, \dots$) becomes constant then $\text{gl.dim}\,(\Lambda/\mathfrak{a}) < \infty$. Thus if Λ is hereditary and semi-primary then $\text{gl.dim}\,(\Lambda/\mathfrak{a}) < \infty$, and we are able to give, in §2, rather precise estimates for $\text{gl.dim}\,(\Lambda/\mathfrak{a})$.

In §3 we show by examples that the above results are the best possible; thus for each pair (m, n) with $0 < m \leqq \infty$, $0 < n \leqq \infty$, $(m, n) \neq (1, \infty)$ we construct a semi-primary ring Λ and a two-sided ideal $\mathfrak{a}$ of Λ such that $\text{gl.dim}\,\Lambda = m$, $\text{gl.dim}\,(\Lambda/\mathfrak{a}) = n$.

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1) Our notion of "semi-primary" does not coincide with "halbprimär" of Deuring, Algebren, Ergebn. Math.