ON THE CHAIN PROBLEM OF PRIME IDEALS

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There is a problem called the chain problem of prime ideals, which asks, when $\mathfrak{o}$ is a Noetherian local integral domain, whether the length of an arbitrary maximal chain of prime ideals in $\mathfrak{o}$ is equal to rank $\mathfrak{o}$ or not.

In the present paper, we want to show that the answer is not affirmative in the general case. On the other hand, we shall show that if the ring $\mathfrak{o}$ is quasi-unmixed (the quasi-unmixedness is a generalized notion of the unmixedness (= equi-dimensionality)), then the answer of the above question is affirmative.

In order to discuss the problem, we first introduce some conditions on chains of prime ideals in a ring in § 1. Then, in § 2, we introduce the notion of quasi-unmixedness of semi-local rings and we shall show that in every quasi-unmixed semi-local ring the chain conditions which will be introduced in § 1 hold (and in particular we see that in every quasi-unmixed semi-local ring, the length of an arbitrary maximal chain of prime ideals is equal to the rank of the ring). In § 3, we shall construct a counter example against the chain problem. In § 4, we shall state some sufficient conditions for a Noetherian local ring to be unmixed.

Terminology. Terms which were used in Nagata [3] and [5] are used in the same sense, except for local or semi-local rings; local or semi-local rings may be non-Noetherian (see [4]).

Results assumed to be known. Besides some results stated in Nagata [3], we need some basic results on local rings (see, for example, [5, § 1]) and some results contained in [5] and [6].

§ 1. Some remarks on chains of prime ideals

Let $\mathfrak{o}$ be a ring. Then we can introduce the following condition in $\mathfrak{o}$:

The first chain condition. Every maximal chain of prime ideals in $\mathfrak{o}$ is of length equal to rank $\mathfrak{o}$.

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