A NOTE ON UNITS OF ALGEBRAIC NUMBER FIELDS

TOMIO KUBOTA

We shall prove in the present note a theorem on units of algebraic number fields, applying one of the strongest formulations, be Hasse [3], of Grunwald's existence theorem.

THEOREM. *Let k be an algebraic number field, l a prime number, E_k the group of units of k and H a subgroup of E_k containing all l-th powers of elements of E_k. Assume that, for every $\eta \in H$, $k(\sqrt[l]{\eta})$ is always ramified over k whenever k contains an l-th root ζ_l ($\neq 1$) of unity. Then there are infinitely many cyclic extentions K/k of degree l with following properties:*

a) $N_{K/k}E_K = H$, where E_K is the group of units of K.

b) if an ideal $\mathfrak{a}$ of k is principal in K, then $\mathfrak{a}$ is principal in k.

Proof. Denote by B the group of elements β of $k^{\times}$ [1] such that (β) is an l-th power of some ideal in k, and denote by $\mathfrak{C}$ the group of ideal classes of k. Let W be the group generated by H and all l-th powers of elements of $k^{\times}$, and let

$$B = B_0 \supset B_1 \supset \ldots \supset B_{s-1} \supset B_s = W \tag{1}$$

be a sequence of subgroups of B such that $(B_{i-1} : B_i) = l$ for every $i\,(1 \leqq i \leqq s)$. As preliminaries, we shall prove that, for every i, there is a prime ideal $\mathfrak{p}_i$ of k which satisfies the following conditions: i) an element γ of B_{i-1} is an l-th power of some element in the $\mathfrak{p}_i$-adic field $k_{\mathfrak{p}_i}$ if and only if γ belongs to B_i. ii) The set of ideal classes of $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ contains an independent base of $\mathfrak{C}/\mathfrak{C}^l$. Assume first that $k \not\ni \zeta_l$. Set $k_l = k(\zeta_l)$. Let $\Lambda = k_l(\sqrt[l]{B})$ be the field obtained from k_l by adjoining all l-th roots of elements of B. Then Λ contains no cyclic extention of degree l over k. For, if L/k is cyclic of degree l, and $L \subset \Lambda$, then k_lL/k is abelian, k_lL/k_l is cyclic of degree l and therefore $k_lL = k_l(\sqrt[l]{\beta})$, where

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[1] We shall use this notation to stand for the multiplicative group of non-zero elements of a field.