ON ASSOCIATIVE COMPOSITIONS IN FINITE NILPOTENT GROUPS

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Let

$$f(X, Y) = X^{m_1} Y^{n_1} \dots X^{m_r} Y^{n_r} \tag{1}$$

be a word in two variables X, Y, i.e. an element in the free group F_2 on two generators X, Y. Let us say that f defines an associative composition for a group G if for arbitrary elements a, b, c in G we have

$$(a \circ b) \circ c = a \circ (b \circ c) \tag{2}$$

where $a \circ b$ is defined by

$$a \circ b = f(a, b). \tag{3}$$

Now Mr. M. Kuranishi raised the following problem: when f defines an associative composition for every group G?

We shall solve this problem in this note (Proposition 1), and determine moreover associative compositions holding for all finite nilpotent groups using a theorem of Prof. K. Iwasawa[1] (Proposition 2). This result will be refined by Proposition 3.

PROPOSITION 1. *In order that $f(X, Y)$ define an associative composition for a free group F_2 on two generators, it is necessary and sufficient that f is one of the following five types:*

$$1,\ X,\ Y,\ XY,\ YX. \tag{4}$$

Proof. An element $t \neq 1$ of a free group generated by x and y can be expressed uniquely in the form $z_1^{e_1} \dots z_k^{e_k}$, where every z_i is either x or y, where $z_i \neq z_{i+1}$ and where e's are non-vanishing integers. k is called the length of t, and is denoted by $l(t)$ (set $l(1) = 0$). Then one will easily verify

$$l(t^f) \geqq l(t), \qquad (f \neq 0), \tag{5}$$

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