ON THE THEORY OF HENSELIAN RINGS, II

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In a previous paper,[1] we studied a general theory of integrally closed Henselian integrity domains and some properties of Henselian valuation rings. The present paper is its continuation. The main aim of the present paper is to study a general theory of Henselian local integrity domains; in the present paper we call a ring $\mathfrak{o}$ a local ring if $\mathfrak{o}$ is a quasi-local ring and if the intersection of all powers of the maximal ideal of $\mathfrak{o}$ is zero, and in this case we introduce a topology by taking the system of all powers of the maximal ideal as a system of neighbourhoods of zero.

Chapter I is concerned mainly with integrally closed local integrity domains: We prove that 1) if a Henselian ring $\mathfrak{h}$ contains an integrally closed quasi-local integrity domain $\mathfrak{o}$, then $\mathfrak{h}$ contains the Henselization of $\mathfrak{o}$, provided that the maximal ideal of $\mathfrak{h}$ lies over that of $\mathfrak{o}$ and 2) if $\mathfrak{o}$ is an integrally closed local integrity domain then its Henselization $\mathfrak{o}^*$ is a local ring which contains $\mathfrak{o}$ as a dense subspace; here, if $\mathfrak{o}$ is Noetherian then so is $\mathfrak{o}^*$, too.

In Chapter II, we prove first the following: Let $\mathfrak{o}$ be an integrally closed quasi-local integrity domain and let $\mathfrak{o}^*$ be its Henselization. If $\mathfrak{p}$ is a prime ideal of $\mathfrak{o}$ then $\mathfrak{p}\mathfrak{o}^*$ is a semi-prime ideal; when $\mathfrak{o}/\mathfrak{p}$ is integrally closed, then $\mathfrak{p}\mathfrak{o}^*$ is a prime ideal. Then we study the nature of $\mathfrak{o}^*/\mathfrak{p}\mathfrak{o}^*$ and study some properties of general Henselian integrity domains.

In our treatment we make use of following two lemmas: 1) If an integrity domain $\mathfrak{o}$ is finitely generated (over a prime integrity domain), then for any prime ideal $\bar{\mathfrak{p}}$ of the integral closure $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ in its quotient field, $\bar{\mathfrak{o}}_{\bar{\mathfrak{p}}}$ is Noetherian.

2) Let $\mathfrak{o}$ be a Noetherian local integrity domain. If the completion of $\mathfrak{o}$ has no nilpotent elements, then the integral closure $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ in its quotient field is a finite $\mathfrak{o}$-module. These lemmas will be discussed in Appendix.

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[1] On the theory of Henselian rings, Nagoya Math. J. 5 (1953), pp. 45–57, which will be referred as [H.R.] in the present paper.