HOMOTOPY CLASSIFICATION OF MAPPINGS OF A 4-DIMENSIONAL COMPLEX INTO A 2-DIMENSIONAL SPHERE

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Steenrod [1] solved the problem[1] of enumerating the homotopy classes of maps of an $(n+1)$-complex K into an n-sphere S^n utilizing the cup-i-product, the far-reaching generalization of the Alexander-Čech-Whitney cup product [7] and the Pontrjagin $*$-product [5].

Since Steenrod's paper [1] appeared, the efforts to extend the result to the case where an $(n-1)$-connected space takes the place of S^n have been made by Whitney [8], Postnikov [10] in case $n=2$. and by Postnikov [11] in case $n>2$.

On the other hand, the $(n+2)$-homotopy group $\pi_{n+2}(S^n)$ of S^n was recently determined to be cyclic of order 2 by Pontrjagin [6], G. W. Whitehead [13]. then an attempt to enumerate the homotopy classes of maps of an $(n+2)$-complex K into S^n is expected.[2]

In the present paper this problem will be solved in case $n=2$. As a partial result as to the n-dimensional case a theorem concerning the third obstruction was obtained (this was announced in our previous note [20] without proof). Let two maps f, g of an $(n+2)$-complex K into S^n be homotopic to each other on the $(n+1)$-skeleton K^{n+1} then there exists a map g' such that g' is homotopic to $g (g' \sim g)$ and $g'=f$ on K^{n+1}, and hence $f^*S^n = g'^*S^n \backsim g^*S^n$ (where S^n is the generating n-cocycle of S^n and f^*, g^* are the cochain homomorphisms induced by f, g). The separation cocycle $d^{n+2}(f, g')$ with coefficients in $\pi_{n+2}(S^n)$ is readily defined. In case $n=2$, $f \sim g$ on K if and only if there exists a 1-cocycle λ^1 of K such that $2f^*S^2 \smile \lambda^1 \backsim 0$ and the cohomology class

$$\{d^4(f, g')\} \equiv \{v_\lambda^2 \smile v_\lambda^2\} \mod S_{q_0}H^2(K, \pi_3(S^2)),$$

where v_λ^2 is a 2-cochain such that $\delta v_\lambda^2 = 2f^*S^2 \smile \lambda^1$. In case $n>2$, a sufficient (not necessary) condition for f, g to be homotopic is obtained:

$$\{d^{n+2}(f, g')\} \equiv 0 \mod S_{q_{n-2}}H^n(K, \pi_{n+1}(S^n)).$$

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1) The problem in case $n=2$ was solved by Pontrjagin [4] and independently by Whitney (an abstract in Bull. Amer. Math. Soc., 42 (1936), p. 338).

2) Problem 15 in Eilenberg. "On the problems of topology," Ann. of Math., 50 (1949), 247-260.