

# ON THE THEORY OF HENSELIAN RINGS

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**Introduction.** The notion of Henselian rings was introduced by G. Azumaya [1].<sup>1)</sup> We concern ourselves in the present paper mainly with Henselizations of integrally closed integrity domains. Chapter I deals with general integrally closed integrity domains. As a preparation of our studies, we introduce the notion of decomposition rings analogously as in the case of fields (§1). And then we define the notions of (local) Henselian rings and Henselizations of integrally closed integrity domains, and obtain several results concerning characterizations of Henselian rings and the uniqueness of Henselizations (§2).

In Chapter II, we restrict ourselves to the case of valuation rings. First we show that although the definition of Henselian rings are concerned with monic polynomials and the maximal ideal, the Hensel lemma holds also for non-monic polynomials (§3) and even modulo not necessarily prime ideals (with certain conditions) (§5).

Appendix (I) gives a proof of a fundamental lemma concerning extensions of a valuation, which is quoted in §3 and Appendix (II) shows an example of a certain type of Henselian, special, discrete valuation ring.

As for the terms, a ring (or an integrity domain) means always commutative one with identity and a ring which has only one maximal ideal is called quasi-local.

We refer to the notations as  $\mathfrak{o}_{\mathfrak{p}}$ , where  $\mathfrak{o}$  is a ring and  $\mathfrak{p}$  is its prime ideal, the ring of quotients of  $\mathfrak{p}$  with respect to  $\mathfrak{o}$ .

## Chapter I.

### General theory of integrally closed Henselian integrity domains.

#### 1. Decomposition rings.

**LEMMA 1.** Let  $\mathfrak{o}$  be an integrally closed integrity domain with quotient field  $K$ . Assume that  $K'$  is a normal (algebraic) extension of  $K$  and let  $\mathfrak{o}'$  be the totality of  $\mathfrak{o}$ -integers in  $K'$ . If  $\mathfrak{p}'_1$  and  $\mathfrak{p}'_2$  are prime ideals in  $\mathfrak{o}'$  such that  $\mathfrak{p}'_1 \cap \mathfrak{o} = \mathfrak{p}'_2 \cap \mathfrak{o}$ , then  $\mathfrak{p}'_1$  and  $\mathfrak{p}'_2$  are conjugate to each other over  $K$ .

*Proof.* When  $K'$  is finite over  $K$ , our proof is easy,<sup>2)</sup> while the general

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<sup>1)</sup> The numbers in brackets refer to bibliography at the end.

<sup>2)</sup> Cf. [5, Theorem 5] or the proof of [6, Lemma 1].