ON THE BEHAVIOR OF AN ANALYTIC FUNCTION ABOUT AN ISOLATED BOUNDARY POINT

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Introduction. Let D be an open set in the z-plane, C its boundary, z_0 a point on C, and $f(z)$ a one-valued meromorphic function in D. Given a set $E \subset D+C$, we denote the intersection of E with $G_r = \{0 < |z-z_0| < r\}$ by E_r, and the set of values $\{f(z); z \in D_r\}$ by $f(D_r)$. The *cluster set* $S_{z_0}^{(D)}$ of $f(z)$ at z_0 in D is defined by $\bigcap_r [f(D_r)]^a$, where $[\ ]^a$ denotes the closure of the set in $[\ ]$, and the *range of values* $R_{z_0}^{(D)}$ is defined by $\bigcap_r f(D_r)$. Further the cluster set $S_{z_0}^{(E)}$ on E is defined by $\bigcap_r [\bigcup_{z \in E_r} S_z^{(D)}]^a$, where $S_z^{(D)}$ at an inner point z is put equal to $f(z)$. In the *theory of cluster sets* relations between $S_{z_0}^{(D)}$, $S_{z_0}^{(C)}$, $R_{z_0}^{(D)}$ are pursued chiefly.[1] Here we refer to the following two principal theorems under the assumption that z_0 is non-isolated:

(I) (Brelot[2]). $(S_{z_0}^{(D)})^b \subset S_{z_0}^{(C)}$, where $(\ )^b$ denotes the boundary of the set in $(\ )$.

(II) (Kunugui [5]). Each component of $S_{z_0}^{(D)} - S_{z_0}^{(C)}$, with two possible exceptions, is contained in $R_{z_0}^{(D)}$, provided that D is a domain.[3]

It is always assumed that z_0 is *non-isolated* in these theorems, and the case when z_0 is isolated is left to the well-known Picard's theorem.

Above the cluster sets are defined for a function which takes values in a plane. However, the definitions can be generalized to a function, which is defined in a plane domain and takes values on an *abstract Riemann surface*, and

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1) For various results and literatures, cf. [7].

2) See [2], Theorem in §6. The form of Brelot's theorem is different from (I), but the equivalency is proved as usual. Cf. [6], for instance.

3) This theorem can be proved also in the case where D is any open set as follows: Suppose that there exists a component Ω of $S_{z_0}^{(D)} - S_{z_0}^{(C)}$, at least three points of which do not belong to $R_{z_0}^{(D)}$. Let w_0 be such an exceptional value. Since $w_0 \in S_{z_0}^{(D)}$, we can choose $\{z_n\}$, $z_n \to z_0$, such that $f(z_n) \to w_0$. Among the inverse images in D of the segments $\overline{f(z_n)w_0}$ in Ω, we can find an inverse image l in D terminating at z_0. $f(z)$ has a limit $w_1 \in \Omega$ as $z \to z_0$ along l. Let D_1 be the component of D which contains l, and C_1 its boundary. Then $S_{z_0}^{(D_1)}$ contains w_1, and $S_{z_0}^{(D)} \supset S_{z_0}^{(D_1)}$, $S_{z_0}^{(C)} \supset S_{z_0}^{(C_1)}$, $R_{z_0}^{(D)} \supset R_{z_0}^{(D_1)}$. The component Ω_1, which contains w_1, of $S_{z_0}^{(D_1)} - S_{z_0}^{(C_1)}$ includes Ω by (I). Hence $R_{z_0}^{(D_1)}$ does not contain at least three values in Ω_1. This is contrary to (II).