A REMARK ON FINITELY GENERATED MODULES

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Theorem 5 of Azumaya's recent article[1] can be formulated in the following generalized form:

I. Let R be a ring. Let $\mathfrak{m}$ be a finitely generated right-module of R such that $\mathfrak{m}R=\mathfrak{m}$. Assume that $\mathfrak{m}=u_1\mathfrak{r}+u_2\mathfrak{r}+ \dots +u_m\mathfrak{r}$ for every generating system $u_1, u_2, \dots, u_m$ of $\mathfrak{m}$ and for every maximal right-ideal $\mathfrak{r}$ of R. Then $\mathfrak{m}=0$.

For the proof, we first consider the case where R possesses a unit element 1. Then the assertion can be proved quite similarly as in Azumaya, l. c. Let namely $u_1, u_2, \dots, u_m$ be any finite generating system of $\mathfrak{m}$; $\mathfrak{m}=u_1R+u_2R+ \dots +u_mR$. Let $\mathfrak{r}_0$ be the right-ideal of R consisting of all elements x of R such that

$$u_1x\in u_2R+ \dots +u_mR.$$

Suppose $\mathfrak{r}_0\neq R$. There exists a maximal right-ideal $\mathfrak{r}$ which contains $\mathfrak{r}_0$, and we have $\mathfrak{m}=u_1\mathfrak{r}+u_2\mathfrak{r}+ \dots +u_m\mathfrak{r}$, whence $=u_1\mathfrak{r}+u_2R+ \dots +u_mR$ much the more, by our assumption. There is an element a in $\mathfrak{r}$ such that $1-a\in\mathfrak{r}_0\subseteq\mathfrak{r}$, which is a contradiction. Hence necessarily $\mathfrak{r}_0=R$ and $\mathfrak{m}=u_2R+ \dots +u_mR$. Now the assertion can be proved by an induction with respect to the minimal number of generating elements.

Let next R be general. Let R^* be the ring which is as module a direct sum of R and the ring of rational integers and in which $1x=x1=x$ $(x\in R)$. If $\mathfrak{r}^*$ is a maximal right-ideal of R^*, then $\mathfrak{r}^*\cap R$ is either R or a maximal right-ideal of R. Thus $\mathfrak{m}=u_1\mathfrak{r}^*+u_2\mathfrak{r}^*+ \dots +u_m\mathfrak{r}^*$, much the more, and the assertion can be reduced to the above case.

It is perhaps of interest to observe that from this generalization Jacobson's theorem[2] may be derived:

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[1] G. Azumaya, On maximally central algebras, Nagoya Math. J. **2** (1951).

[2] N. Jacobson, The radical and semi-simplicity for arbitrary rings, Amer. J. Math. **67** (1945), Theorem 10; the present formulation is in 1), l. c.