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Nagoya Math. J.
Vol. 44 (1971), 51–55

UNITS OF REAL QUADRATIC FIELDS

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1. Let D be a positive square-free integer. Throughout this note we shall use the following notations;

$d = d(D)$: the discriminant of $\boldsymbol{Q}(\sqrt{D})$,

t_0, u_0 : the least positive solution of Pell's equation $t^2 - du^2 = 4$,

$\varepsilon_D = (t_0 + u_0\sqrt{d})/2$.

In this note we estimate ε_D. At first (in lemma) we prove that for $\boldsymbol{Q}(\sqrt{D})$ there exist integers ℓ, m and Δ (= square-free) such that D is one of three types

$$D = \Delta\Big(m^2\Delta \pm \frac{4}{2^\delta}\Big)\Big/\ell^2, \qquad (\delta = 0,\ 1 \text{ or } 2)$$

where $2 \nmid m$, $2 \nmid \Delta$ for $\delta = 0$ and $2 \nmid \Delta$ for $\delta = 1$. Therefore we consider the above three types.

As for the estimate of ε_D Hua [1] proved

$$\log \varepsilon_D < \sqrt{d}\Big(\frac{1}{2} \log d + 1\Big). \tag{1}$$

Here we estimate ε_D in accordance with the above three types.

THEOREM. *We have*

$$\varepsilon_D < 2^\delta \ell^2 D, \tag{2}$$

where $D = \Delta(m^2\Delta + 4/2^\delta)/\ell^2$ *and* $\delta = 0$, 1 *or* 2. Δ *is a square-free integer* > 0, m *and* ℓ *are integers. In particular* $2 \nmid m$, $2 \nmid \Delta$ *for* $\delta = 0$ *and* $2 \nmid \Delta$ *for* $\delta = 1$. *More precisely when* $\delta = 1$ *we have*

$$\varepsilon_D < \begin{cases} 2\ell^2 D & (\Delta = 1), \\ \ell^2 D & (\Delta \geqslant 2), \end{cases} \tag{3}$$

Received January 25, 1971