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Nagoya Math. J.
Vol. 43 (1971), 199–208

# ON ELLIPTIC CURVES WITH COMPLEX MULTIPLICATION AS FACTORS OF THE JACOBIANS OF MODULAR FUNCTION FIELDS

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**1.** As Hecke showed, every $L$-function of an imaginary quadratic field $K$ with a Grössen-character $\lambda$ is the Mellin transform of a cusp form $f(z)$ belonging to a certain congruence subgroup $\Gamma$ of $SL_2(\boldsymbol{Z})$. We can normalize $\lambda$ so that

$$\lambda((\alpha)) = \alpha^{\nu} \quad \text{for} \quad \alpha \in K, \ \alpha \equiv 1 \ \mathrm{mod}^{\times} \mathfrak{c}$$

with a positive integer $\nu$, where $\mathfrak{c}$ is the conductor of $\lambda$, and $\mathrm{mod}^{\times}\,\mathfrak{c}$ means the multiplicative congruence modulo $\mathfrak{c}$. Then $f(z)$ is of weight $\nu+1$, i.e.,

$$f((az+b)/(cz+d)) = f(z)\,(cz+d)^{\nu+1} \quad \text{for} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma,$$

and $\Gamma$ is given by

$$\Gamma = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\boldsymbol{Z}) \,\middle|\, a \equiv d \equiv 1, \ c \equiv 0 \ \mathrm{mod}\ (D \cdot N(\mathfrak{c})) \right\},$$

where $-D$ is the discriminant of $K$. If $\nu = 1$, $f(z)dz$ is a differential form of the first kind on the compactification $(H/\Gamma)^*$ of the quotient $H/\Gamma$, where $H$ denotes the upper half complex plane. Denote by $\mathrm{Jac}(H/\Gamma)$ the jacobian variety of $(H/\Gamma)^*$, and identify the tangent space of $\mathrm{Jac}(H/\Gamma)$ at the origin with the space of all differential forms of the first kind on $(H/\Gamma)^*$. Let $A$ be the smallest abelian subvariety of $\mathrm{Jac}(H/\Gamma)$ that has $f(z)dz$ as a tangent at the origin. Then the first main result of this paper can be stated as follows:

*The abelian variety $A$ is a product of copies of an elliptic curve whose endomorphism algebra is isomorphic to $K$.*

Hecke [3] proved this fact in the case where $K = \boldsymbol{Q}(\sqrt{-q})$ with a prime $q > 3$, $\equiv 3 \bmod (4)$ and $\mathfrak{c} = (\sqrt{-q})$. In the general case, he showed only that

Received January 16, 1971.