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A GENERALIZATION OF EPSTEIN'S ZETA FUNCTION

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§ 0. Introduction.

Koecher defined in [3] the following zeta function associated with the matrix $S^{(n)}$ of a positive quadratic form and one complex variable ρ

$$Z_{n_1}(S, \rho) = \sum_A |{}^tASA|^{-\rho}. \tag{1}$$

Here $n \geqq n_1$, and the sum is over a complete set of representatives for the n by n_1 integral rank n_1 matrices $A^{(n,n_1)}$ with respect to the equivalence relation $A \sim B$ if $A = BU$ for some unimodular matrix U. The unimodular group $\mathfrak{U}_{n_1}$ is defined by $\mathfrak{U}_{n_1} = \{U^{(n_1)}: U$ integral, n_1 by n_1 with determinant $|U| = \pm 1\}$. We use the notation $|S|$ = determinant of S and $S[A] = {}^tASA$ throughout. Superscripts in parentheses on matrices denote the number of rows and columns. Thus $A^{(n,n_1)}$ has n rows and n_1 columns.

Koecher shows in [3] that $Z_{n_1}(S, \rho)$ converges for $Re\rho > \frac{n}{2}$. But his proof of the analytic continuation and the functional equation,

$$R_{n_1}(S, \rho) = |S|^{\frac{-n_1}{2}} R_{n_1}\left(S^{-1}, \frac{n}{2} - \rho\right) \tag{2}$$

where $R_{n_1}(S, \rho) = \pi^{n_1\left(\frac{n_1-1}{4} - \rho\right)} \prod_{i=0}^{n_1-1} \Gamma\left(\rho - \frac{i}{2}\right) Z_{n_1}(S, \rho)$,

has a gap. This is remedied neatly using an idea of Selberg. One can annihilate the trouble-making terms of the theta function with an appropriate differential operator. We outline these results in § 1 because they do not appear in the literature.

Selberg has defined in [6] a zeta function associated with a positive matrix S and $n-1$ complex variables $\rho = (\rho_1, \rho_2, \cdots, \rho_{n-1})$. This function can be seen to be essentially the same as

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