AN INTEGRAL FORMULA FOR THE CHERN FORM OF A HERMITIAN BUNDLE

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Introduction

We shall consider a Hermitian n-vector bundle E over a complex manifold X. When X is compact (without boundary), S.S. Chern defined in his paper [3] the Chern classes (the basic characteristic classes of E) $\hat{C}_i(E)$, $i = 1, \cdots, n$, in terms of the basic forms Φ_i on the Grassmann manifold $H(n, N)$ and the classifying map f of X into $H(n, N)$. Moreover he proved ([3], [4]) that if E_k denotes the k-general Stiefel bundle associated with E, the $(n-k+1)$-th Chern class $\hat{C}_{n-k+1}(E)$ coincides with the characteristic class $C(E_k)$ of E_k defined as follows: Let K be a simplicial decomposition of X and $K^{2(n-k)+1}$ the $2(n-k)+1-$shelton of K. Then there exists a section s of $E_k | K^{2(n-k)+1}$ so that one can define the obstruction cocycle $c(s)$ of s. The cohomology class of $c(s)$ is independent of such a section s. Thus one denotes by $C(E_k)$ the cohomology class of $c(s)$ which is called the characteristic class of E_k. The above fact is well known as the second definition of the Chern classes ([3]).

On the other hand, in case when X is with boundary, R. Bott and S.S. Chern established the so-called Gauss-Bonnet theorem ([1]), which gives an integral formula for the above second definition of the n-th Chern class $\hat{C}_n(E)$, that is, if $C_n(E)$ denotes the n-th Chern form induced by a norm on E (c.f. Prop. 2.1),

$$\int_X C_n(E) = \int_{\partial X} s^*\eta_n + \textstyle\sum_{j=1}^{l} \text{zero}\,(p_j;\, s),$$

where the p_j are the zero points of a section s of X into E, the zero $(p_j;\, s)$ denote the zero-numbers of s at p_j, and η_n is the n-th boundary form of E (cf. Def. 3.1).

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