Yoshiyuki Kitaoka
Nagoya Math. J.
Vol. 42 (1971), 89–93

ON A SPACE OF SOME THETA FUNCTIONS

YOSHIYUKI KITAOKA

In the theory of modular forms there is an interesting problem whether every modular form can be expressed as a linear comination of theta functions. For this Eichler proved in [1] that for a sufficiently large prime q all modular forms of degree $-2m(m = 1, 2, \cdots)$ for $\Gamma_0(q)$ can be represented by linear combinations of theta functios of degree $-2m$ with level 1 and q. We prove this theorem for $q = 2$, 3, 5 and 11 by using a theorem of Siegel for $q = 2$, 3, 5 and a general result of Eichler for $q = 11$. The former method is shown in Schoeneberg [2].

Before our statement, it should be recalled: for an even positive $4m \times 4m$ matrix Q with level N and square discriminant, the theta function

$$\vartheta(\tau, Q) = \sum_{\xi \in Z^{4m}} e^{\pi i {}^t\xi Q \xi \tau}$$

is a modular form of degree $-2m$ for $\Gamma_0(N)$, i.e. of type $(-2m, N, 1)$ in the sense of Hecke.

THEOREM. *For $q = 2$, 3, 5 and 11 all modular forms of degree $-2m(m=1, 2, \cdots)$ for $\Gamma_0(q)$ can be represented by linear combinations of theta functions of type $(-2m, q, 1)$ and $(-2m, 1, 1)$.*

Proof for $q = 2$. Let d_m (resp. e_m) be the dimension of the space $\mathfrak{M}(m)$ (resp. $\mathfrak{S}(m)$) of modular forms (resp. cusp forms) of degree $-2m$ for $\Gamma_0(2)$. Then it is well known that

$$(1) \qquad \begin{cases} d_m = \left[\dfrac{m}{2}\right] + 1 & \text{for} \quad m \geq 1, \\ e_m = 0 & \text{for} \quad m = 1, \\ e_m = \left[\dfrac{m}{2}\right] - 1 & \text{for} \quad m \geq 2. \end{cases}$$

Let A be an even positive 4×4 matrix with level 2 and determinant 4, for

Received May 4, 1970.