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KSO-GROUPS FOR 4-DIMENSIONAL CW-COMPLEXES

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§ 0. In this paper we shall determine *KSO*-groups for 4-dimensional *CW*-complexes by their cohomology rings. We denote by $KSO(X)$ the group of orientable stable vector bundles over X. In 1959 A. Dold and H. Whitney [1] gave the classification of $SO(n)$-bundles over a 4-complex. It seems, however, to the authors that group structures of them are unknown. We shall give another definition of the difference bundles defined in [1], and we determine the group structure of $KSO(X)$.

§ 1. For a finite 4-dimensional *CW*-complex X, we denote by X_3 its 3-skeleton, by X/X_3 a complex obtained from X by contracting X_3 to a point in X, and by EX_3 the suspension of X_3. The following exact sequence is obtained from Puppe's sequence.

$$\text{(I)}\qquad \longrightarrow KSO(EX_3) \xrightarrow{j^*} KSO(X/X_3) \xrightarrow{p^*} KSO(X) \xrightarrow{i^*} KSO(X_3) \longrightarrow 0.$$

At first we define a map $W_k\colon KSO(X) \longrightarrow H^k(X; Z_2)$ which assigns to each bundle over X its k-th Whitney class. The following lemma is well known.

LEMMA 1-1. *The homomorphism* $W_2\colon KSO(X_3) \longrightarrow H^2(X_3; Z_2)$ *is an isomorphism.*

Secondly we define a map $P_1\colon KSO(X) \longrightarrow H^4(X; Z)$ which assigns to each element of $KSO(X)$ its first Pontrjagin class. Then we have

LEMMA 1-2.[1)] *For any finite CW-complex* X, *the map* $P_1\colon KSO(X) \longrightarrow H^4(X; Z)$ *is a group homomorphism.*

Proof. If ξ and η are orientable stable vector bundles over X, we can take $\tilde{\xi}\colon X \longrightarrow BSO(m)$ and $\tilde{\eta}\colon X \longrightarrow BSO(n)$ as their classifying maps for

1) This lemma and its proof are suggested to the authors by the referee, and the original lemma was proved under the condition that dim $X \leqq 4$.