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ON THE FINITE SUBGROUPS OF GL (3, Z)

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Introduction

We should like to study three dimensional algebraic tori in the same way as Voskresenskiĭ does in [14] and [15]. To do so, it is necessary to determine all finite subgroups of $GL(3, \boldsymbol{Z})$ up to conjugacy.

We find in Serre [11] that the order of any finite subgroup of $GL(3, \boldsymbol{Z})$ is at most $N(n)$, where $N(n)$ is the greatest common divisor of $2^{n^2}(2^n-1)(2^n-2)\cdots\cdot(2^n-2^{n-1})$ and $(p^n-1)(p^n-p)\cdot\cdots\cdot(p^n-p^{n-1})$ for every odd prime p. According to Serre himself*), this estimate was first obtained by Minkowski [16]. This estimate, however, is not the best possible. For example, when $n=2$, the greatest of the orders of all finite subgroups is $2^2\cdot 3=12$ (cf. Serre, ibid.), while $N(n)=48$. We refer the reader to a sharper estimate of the orders of all finite subgroups of $GL(n, \boldsymbol{Z})$ by Minkowski [17]. According to this, the greatest is not larger than $2^4\cdot 3=48$ when $n=3$. In this paper we show that this is the best possible, and further determine all the finite subgroups of $GL(3, \boldsymbol{Z})$ (resp. $SL(3, \boldsymbol{Z})$) up to conjugacy.

First of all, we find all non-conjugate cyclic subgroups of $GL(3, \boldsymbol{Z})$. By Vaidyanathaswamy [12] and [13], any element of $GL(3, \boldsymbol{Z})$ has order 1, 2, 3, 4, 6 or ∞: namely $\varphi(m)\leq 2$ only for $m=1$, 2, 3, 4 or 6, where $\varphi(m)$ is Euler's function. Hence the order of any finite cyclic subgroup of $GL(3, \boldsymbol{Z})$ is 1, 2, 3, 4, or 6. Reiner [10] determined all non-conjugate cyclic subgroups of order m in $GL(3, \boldsymbol{Z})$ for prime numbers $m=2$ and 3. Therefore we must determine all non-conjugate cyclic subgroups of order m in $GL(3, \boldsymbol{Z})$ for $m=4$ and 6.[1)]

Next we determine all non-conjugate non-cyclic subgroups of $GL(3, \boldsymbol{Z})$. Since each element of $GL(3, \boldsymbol{Z})$ has order 1, 2, 3, 4, 6 or ∞, the order of any

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1) For $m=6$, *see* Matuljauskas [7].