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ON A PROBLEM OF DOOB CONCERNING MULTIPLY SUPERHARMONIC FUNCTIONS

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The following is a well-known result due to A.P. Calderon [2], asserting the existence of non-tangential limits of multiply harmonic functions.

Let $E = E_1 \times E_2 \times \cdots \times E_m$ be the cartesian product of the spaces E_k of points $P_k(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$, and $F(P)$, $P = (P_1, \cdots, P_m) \in E$, be defined and continuous in $x_n^{(k)} > 0$, $k = 1, 2, \cdots, m$, and harmonic in P_k, that is, such that

$$\sum_{i=1}^{n} \frac{\partial^2 F}{(\partial x_i^{(k)})^2} = 0 \quad k = 1, 2, \cdots, m.$$

Let $B_k \subset E_k$ be the space $x_n^{(k)} = 0$, and $B = B_1 \times B_2 \times \cdots \times B_m$ the so-called distinguished boundary of $x_n^{(k)} > 0$, $k = 1, 2, \cdots, m$, and suppose that for every point $Q = (Q_1, Q_2, \cdots, Q_m)$, $Q_i \in B_i$, of a set A of positive measure of B, there exist regions Γ_{kQ}, limited by cones with vertices at the points Q_k and hyperplanes $x_n^{(k)} = \text{const}$ such that the function $F(P)$ is bounded in $\Gamma_Q = \Gamma_{1Q} \times \Gamma_{2Q} \times \cdots \times \Gamma_{mQ}$. Then almost everywhere in A, $F(P)$ has a limit as $P = (P_1, \cdots, P_m)$ tends to $Q = (Q_1, \cdots, Q_m) \in A$ in such a way that all P_k tend to Q_k simultaneously and non-tangentially.

Generalizing the above result in the case of functions of one variable, but on Green spaces, J.L. Doob [4] proved the following.

Let Ω be a Green space and Δ its Martin boundary. Let u and h be two superharmonic functions on Ω, $h > 0$. If, for every $z \in E \subset \Delta$, $\frac{u}{h}$ is bounded below in a set which is not thin at z, then $\frac{u}{h}$ has a finite fine limit at μ_h almost every point of E; where μ_h is the canonical measure corresponding to h in the Riesz-Martin integral representation with measures on $\Omega \cup \Delta$.

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