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Nagoya Math. J.
Vol. 39 (1970), 39–79

A CHARACTERIZATION OF THE FINITE SIMPLE GROUPS PSp(4,q), $G_2(q)$, $D_4^2(q)$, II

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Our object in this paper is to prove the following result.

THEOREM. *Let G be a finite group satisfying the following conditions:*

(*) *G has subgroups L_1, L_2 such that $L_1 \simeq SL(2, q_1)$, $L_2 \simeq SL(2, q_2)$, $[L_1, L_2]=1$, $L_1 \cap L_2 = \langle j \rangle$, where j is an involution, and $|C(j) : L_1L_2| = 2$.*

(**) *$C(j) = L_1L_2\langle n\rangle$, where $n^2 = 1$, $L_1^n = L_1$, $L_2^n = L_2$.*

Then $G = C(j)O(G)$, or G is isomorphic to one of the simple groups $G_2(q)$ or $D_4^2(q)$, where $q = min\{q_1, q_2\}$.

The groups $G_2(q)$ are the simple groups of order $q^6(q^6-1)(q^2-1)$ discovered by Dickson [3], [4] in the 1900's. The groups $D_4^2(q)$ are the simple groups of order $q^{12}(q^6-1)(q^2-1)(q^8+q^4+1)$ discovered by Steinberg and Tits [8], [13] in the 1950's. These groups, for q odd, thus take their place among those finite simple groups which can be characterized by the structure of the centralizer of an involution.

Some remarks on the theorem and its proof may be appropriate at this point. Condition (**) can be dropped if G is assumed to be not isomorphic with $PSp(4,q)$, where $q = \min\{q_1, q_2\}$. This is a consequence of [5] (2*A*) and [15]. Moreover, [5] (7*I*) implies that either q_1 and q_2 are equal, or one is the cube of the other, these being in fact the values of the parameters q_1, q_2 in case G is $G_2(q)$ or $D_4^2(q)$. If $(q_1q_2)^3$ is assumed to divide $|G|$, then it is fairly straightforward to construct a subgroup $\tilde{G}$ of G which is isomorphic to $G_2(q)$ or $D_4^2(q)$. This is accomplished by presenting $\tilde{G}$ as a group with a (B,N)-pair in the sense of Tits [12] and imposing a unique multiplication table on B and on N, and hence on $\tilde{G}$. $\tilde{G}$ can then be

Received January 16, 1969.

1) This research was partially supported by the National Science Foundation grant *GP*-6539.