M. Nagata
Nagoya Math. J.
Vol. 37 (1970), 191–196

ON SELF-INTERSECTION NUMBER OF A SECTION ON A RULED SURFACE

MASAYOSHI NAGATA

To Professor K. Ono for his sixtieth birthday

Let E be a non-singular projective curve of genus $g \geq 0$, $\boldsymbol{P}$ the projective line and let F be the surface $E \times \boldsymbol{P}$. Then it is well known that a ruled surface F^* which is birational to F is biregular to a surface which is obtained by successive elementary transformations from F (for the notion of an elementary transformation, see [3]). The main purpose of the present article is to prove the following

THEOREM 1. *For any such F^*, there is a section (i.e., an irreducible curve s on F such that $(s, l) = 1$ for a fibre l of F^*) such that its self-intersection number (s, s) is not greater than g.*

In classifying ruled surface F^*, as was noted by Atiyah [1], it is important to know the minimum value of self-intersection numbers (s, s) of sections of F^*.[1)] Our Theorem 1 is important in the respect.

The following is a key to our proof of Theorem 1:

THEOREM 2. *Let d be a non-negative rational integer. If $Q_1, \cdots, Q_{g+2d+1}$ are points[2)] of F, then there is a positive divisor D of F such that* (i) *D goes through $Q_1, \cdots, Q_{g+2d+1}$ and* (ii) *D is linearly equivalent to $E \times P + \sum_{i=1}^{g+d} R_i \times \boldsymbol{P}$ with a $P \in \boldsymbol{P}$ and suitable $R_i \in E$.*

In connection with this Theorem 2, we prove the following theorem too:

Received October 11, 1968

1) Atiyah proved that the minimum value is not greater than $2g-1$ if $g>0$. On the other hand, it was remarked by M. Maruyama that there is an F (for every E) which carries only sections s such that $(s, s) \geqq g$ (see [2]).

2) In this theorem, these Q_i need not be ordinary points, namely, some of these Q_i may be infinitely near points of some ordinary points. For the definition of the term "go through" in such a case, see [3].