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A REMARK ON PEIRCE'S LAW

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Before stating the purpose, we explain some propositional logics treated in this paper. The logical symbols we use are: implication $\rightarrow$, conjunction $\wedge$, disjunction $\vee$, and the propositional constant $\curlywedge$ denoting contradiction. The axioms for the intuitionistic propositional logic (denoted by ***LJS***) are:

(I) $p \rightarrow (q \rightarrow p)$, $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$,

(C) $(p \wedge q) \rightarrow p$, $(p \wedge q) \rightarrow q$, $(r \rightarrow p) \rightarrow ((r \rightarrow q) \rightarrow (r \rightarrow (p \wedge q)))$,

(D) $p \rightarrow (p \vee q)$, $q \rightarrow (p \vee q)$, $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r))$,

(F) $\curlywedge \rightarrow p$.

The rules of inference are *modus ponens* and *substitution*. The system characterized by the axioms (I) we call the primitive propositional logic (denoted by ***LOS***), which is the propositional part of the primitive logic ***LO*** introduced in Ono [3]. ***LOS*** is also known as the positive implicational logic. The axioms (I), (C), (D) characterize the (full) positive propositional logic (denoted by ***LPS***). Not all classically true formulas expressible in ***LOS*** are derivable from (I); they are derivable from (I) together with the axiom known as *Peirce's law*:

(P) $((p \rightarrow q) \rightarrow p) \rightarrow p$.

The axioms (I), (C), (D), (P) are sufficient for the derivation of all classically true formulas expressible in ***LPS***. Moreover, the axioms (I), (C), (D), (F), (P) characterize the classical propositional logic (denoted by ***LKS***). Indeed, all classically true propositional formulas are provable in ***LKS***. Finally, by deleting the axiom (F) from ***LJS***, we obtain Johansson's minimal propositional logic (denoted by ***LMS***). It is easy to see that the following formula

(M) $((p \rightarrow \curlywedge) \rightarrow p) \rightarrow p$

is equivalent to *the law of the excluded middle* in ***LMS***. Furthermore, it

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