UPPER BOUNDS ON HOMOLOGICAL DIMENSIONS

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The homological dimension of a module M_R is often related to the cardinality of a set of generators for M or for right ideals of R. In this note, upper bounds for this homological dimension are obtained in two situations.

In [8] Jensen has shown that, for any ring R whose finitely generated right ideals are countably related, if any right ideal of R is generated by $\aleph_n$ elements, then the right global dimension of R exceeds the weak global dimension by at most $n+1$. In section 1 we show that the condition that finitely generated right ideals are countably related may be deleted, and Jensen's theorem will still hold.

In [3] Berstein showed that a direct limit of modules over a countable directed system has dimension at most one more than the supremum of the dimensions of the modules. This is also an immediate consequence of Roos [14], Theorem 1. In section 2 we show that a direct limit of modules over a directed system of cardinality $\aleph_n$ has dimension at most $n+1$ more than the supremum of the dimensions of the modules. Balcerzyk showed this for a directed union in [2].

All rings R will have identity 1 ; all modules will be unital right R-modules. For a module M, $hd_R(M)$ (or $hd(M)$ if no confusion arises) will denote the homological dimension of M. gl. d(R) will denote the right global dimension of R and w. gl. d(R) its weak global dimension. A basic tool for calculating upper bounds on homological dimensions is the following proposition of Auslander.

PROPOSITION 0. 1. *Let $\mathscr{I}$ be a non-empty well-ordered set, M a right R-module, $\{N_i \mid i \in \mathscr{I}\}$ a family of submodules of M such that $N_i \subseteq N_j$ for $i \leq j$. If $M = \cup_{i \in \mathscr{I}} N_i$ and $hd_R(N_i / \cup_{j<i} N_j) \leq n$ for all $i \in \mathscr{I}$, then $hd_R(M) \leq n$.*

Proof. This is proposition 3 of [1].

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