ON SQUARE INTEGRABLE MARTINGALES

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To Professor Kiyoshi Noshiro on the occasion of his 60th birthday.

§ 0. Introduction.

Theory of real and time continuous martingales has been developed recently by P. Meyer [8, 9]. Let $\{X_t, \mathfrak{F}_t\}$ be a square integrable martingale on a probability space P. He showed that there exists an increasing process $\langle X \rangle_t$ such that

$$E((X_t - X_s)^2/\mathfrak{F}_s) = E(\langle X \rangle_t/\mathfrak{F}_s) - \langle X \rangle_s \quad a.e.\ P \ for\ t > s > 0.$$

The above formula suggests us that some results concerning Brownian motion can be generalized to those of martingale. Actually, if $\langle X \rangle_t \equiv t$, X_t is a Brownian motion (See [2, Theorem 11. 9], also Theorem 2.3 of this paper), and in general, many continuous martingales may be obtained by time changes of Brownian motions ([3] and Theorem 3.1 of this paper).

Stochastic integral concerning martingale was defined by Meyer [9], Courrège [1] and in some special case, by Motoo and S. Watanabe [10]. In the present paper we shall discuss a formula on stochastic integral which is a generalization of Itô's formula [4] concerning Brownian motion. Let X_t be a Brownian motion and f be a C^2-class function. Itô's formula is the following:

$$f(X_t) - f(X_0) = \int_0^t \frac{df}{dx}(X_s) dX_s + \frac{1}{2} \int_0^t \frac{d^2 f}{dx^2}(X_s) ds.$$

We shall show in § 2 that when X_t is a continuous martingale, the above formula is still valid if ds is replaced by $d\langle X \rangle_s$. When X_t is not continuous, the formula becomes a more complicated form (See § 5). There, Lévy system introduced by one of the authors [11] plays an important role.

The formula on stochastic integral will be applied to two problems. In § 6, we shall discuss the structure of multiplicative functionals of a Markov process. Roughly speaking, every multiplicative functional with mean 1 is factorized into two mutually orthogonal martingales; one is continuous multiplicative functional and the other is a jump type one. § 7 is devoted to giving another proof of Lévy-Itô's decomposition of additive process (process with independent