C. P. Boyer, E. G. Kalnins, and W. Miller, Jr.
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SYMMETRY AND SEPARATION OF VARIABLES FOR THE HELMHOLTZ AND LAPLACE EQUATIONS

C. P. BOYER, E. G. KALNINS, AND W. MILLER, JR.

Introduction.

This paper is one of a series relating the symmetry groups of the principal linear partial differential equations of mathematical physics and the coordinate systems in which variables separate for these equations. In particular, we mention [1] and paper [2] which is a survey of and introduction to the series. Here we apply group-theoretic methods to study the separable coordinate systems for the Helmholtz equation.

$$(\Delta_3 + \omega^2)\Psi(\boldsymbol{x}) = 0\,, \qquad \boldsymbol{x} = (x_1, x_2, x_3)\,, \\ \Delta_3 = \partial_{11} + \partial_{22} + \partial_{33}\,, \qquad \omega > 0\,, \tag{0.1}$$

and the Laplace equation

$$\Delta_3\Psi(\boldsymbol{x}) = 0\,. \tag{0.2}$$

It is well-known that (0.1) separates in eleven coordinate systems, see [3], Chapter 5, and references contained therein. Moreover, in [4] it is shown that these systems correspond to commuting pairs of second order symmetric operators in the enveloping algebra of $\mathscr{E}(3)$, the symmetry algebra of (0.1). However, we show here for the first time how one can systematically make use of the representation theory of the Euclidean symmetry group $E(3)$ of the Helmholtz equation to derive identities relating the different separable solutions. As we will point out, some of these identities are new.

It is also known that there are 17 types of cyclidic coordinate systems which permit R-separation of variables in the Laplace equation and these appear to be the only such separable systems for (0.2), [5]. (An R-separable coordinate system $\{u, v, w\}$ for (0.2) is a coordinate system which permits a family of solutions

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