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# ON THE HOPF FIBRATION $S^7 \to S^4$ OVER $Z$

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## § 1. Statement of the result

Let $K$ be the classical quaternion field over the field $\boldsymbol{Q}$ of rational numbers with the quaternion units $1, i, j, k$, with relations $i^2 = j^2 = -1$, $k = ij = -ji$. For a quaternion $x \in K$, we write its conjugate, trace and norm by $\bar{x}, Tx$ and $Nx$, respectively. Put

$$A = K \times K\,, \qquad B = \boldsymbol{Q} \times K$$

and consider the map $h: A \to B$ defined by

$$h(z) = (Nx - Ny, 2\bar{x}y)\,, \qquad z = (x, y) \in A\,. \tag{1.1}$$

The map $h$ is the restriction on $\boldsymbol{Q}^8$ of the map $\boldsymbol{R}^8 \to \boldsymbol{R}^5$ which induces the classical Hopf fibration $S^7 \to S^4$ where each fibre is $S^3$.[1] For a natural number $t$, put

$$S_A(t) = \{z = (x, y) \in A,\ Nx + Ny = t\}\,, \tag{1.2}$$

$$S_B(t) = \{w = (u, v) \in B,\ u^2 + Nv = t\}\,. \tag{1.3}$$

Then, $h$ induces a map

$$h_t: S_A(t) \to S_B(t^2)\,. \tag{1.4}$$

Now, let $\mathfrak{o}$ be the unique maximal order of $K$ which contains the standard order $\boldsymbol{Z} + \boldsymbol{Z}i + \boldsymbol{Z}j + \boldsymbol{Z}k$. As is well-known, $\mathfrak{o}$ is given by

$$\mathfrak{o} = \boldsymbol{Z}\rho + \boldsymbol{Z}i + \boldsymbol{Z}j + \boldsymbol{Z}k\,, \qquad \rho = \tfrac{1}{2}(1 + i + j + k)\,.$$

The group $\mathfrak{o}^\times$ of units of $\mathfrak{o}$ is a finite group of order 24. The 24 units are: $\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)$. We know that the number of quaternions in $\mathfrak{o}$ with norm $n$ is equal to $24 s_0(n)$ where $s_0(n)$ denotes the sum of odd divisors of $n$.

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1) H. Hopf, Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension, Fund. Math. 25 (1935) 427–440.