T. E. Cecil
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GEOMETRIC APPLICATIONS OF CRITICAL POINT THEORY TO SUBMANIFOLDS OF COMPLEX PROJECTIVE SPACE

THOMAS E. CECIL

Section 0—Introduction.

In a recent paper, [6], Nomizu and Rodriguez found a geometric characterization of umbilical submanifolds $M^n \subset \boldsymbol{R}^{n+p}$ in terms of the critical point behavior of a certain class of functions L_p, $p \in \boldsymbol{R}^{n+p}$, on M^n. In that case, if $p \in \boldsymbol{R}^{n+p}$, $x \in M^n$, then $L_p(x) = (d(x,p))^2$, where d is the Euclidean distance function.

The result of Nomizu and Rodriguez can be expressed as follows. Let M^n $(n \geqslant 2)$ be a connected, complete Riemannian manifold isometrically immersed in $\boldsymbol{R}^{n+p}$. Suppose there exists a dense subset D on $\boldsymbol{R}^{n+p}$ such that every function of the form L_p, $p \in D$, has index 0 or n at any of its non-degenerate critical points. Then M^n is an umbilical submanifold, that is M^n is embedded in $\boldsymbol{R}^{n+p}$ as a Euclidean subspace, $\boldsymbol{R}^n$, or a Euclidean n-sphere, S^n.

Since the set of all points $p \in \boldsymbol{R}^{n+p}$ such that L_p is a Morse function is a dense subset of $\boldsymbol{R}^{n+p}$, the above theorem could also have been stated in terms of Morse functions of the form L_p.

In this paper, we prove results analogous to those of Nomizu and Rodriguez for submanifolds of complex projective space, $P^m(\boldsymbol{C})$, endowed with the standard Fubini-Study metric.

Let M^n be a complex n-dimensional submanifold of $P^{n+p}(\boldsymbol{C})$. For $p \in P^{n+p}(\boldsymbol{C})$, $x \in M^n$, the function $L_p(x)$ which we define is essentially the distance in $P^{n+p}(\boldsymbol{C})$ from p to x. In section 2, we define the concept of a focal point of (M^n, x). We then prove an Index Theorem for L_p which states that the index of L_p at a non-degenerate critical point x is equal to the number of focal points of (M^n, x) on the geodesic in $P^{n+p}(\boldsymbol{C})$ from x to p.

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