Y. N. Nakai
Nagoya Math. J.
Vol. 52 (1973) 163–172

ON A θ-WEYL SUM

YOSHINOBU NAKAI

0°. We treat the sum $\theta(\alpha^{-1}, \gamma; N, X) \underset{\text{def.}}{=} \sum_{X \leqq n \leqq X+N} e((2\alpha)^{-1}(n+\gamma)^2)$, where α and γ are real with α positive.*) This sum was treated first by Hardy and Littlewood [4], and after them, by Behnke [1] and [2], Mordell [9], Wilton [11] and others. The reader will find its history in [7] and in the comments of the Collected Papers [4]. Here we show that the sum can be expressed explicitly, together with an error term $O(N^{1/2})$, using the regular continued fraction expansion of α. As the statements have complications we will divide them into two theorems. In the followings all letters except $\vartheta, i, \sigma, \zeta, \chi$ and those in 3° are real, N is a positive real, and always k, n, a, A, B, C, D and E denote integers. The author expresses his thanks to Professor Tikao Tatuzawa and Professor Tomio Kubota for their encouragements.

1°. LEMMA 1. *Let α, α', γ and γ' be reals such that*

$$\alpha^{-1} \equiv \alpha'^{-1} \qquad \text{mod. } 1$$

and

$$(2\alpha)^{-1}(1+2\gamma) \equiv (2\alpha')^{-1}(1+2\gamma') \qquad \text{mod. } 1\,,$$

then we have

$$(2\alpha)^{-1}(n+\gamma)^2 \equiv (2\alpha')^{-1}(n+\gamma')^2 + (2\alpha)^{-1}\gamma^2 - (2\alpha')^{-1}\gamma'^2 \qquad \text{mod. } 1$$

for any integer n.

Proof. It is easy.

LEMMA 2 (Hardy-Littlewood, Mordell and Wilton). *If* $0 < \omega \leq 2$,

Received June 26, 1973.

*) In this note $e(\alpha)$ means $e^{2\pi i\alpha}$ for real α. $\boldsymbol{N}$ is the set of positive integers. $\boldsymbol{Z}$ is the set of all integers. The implied positive numerical constants in the symbol "$\ll$" in the statements and proofs of (Case 2) of Theorem 1 can be given arbitrarily. Other implied constants in the symbols "$\ll$", "$O(\ )$" and "$\underset{\cap}{\cup}$" are absolute or can be explicitly calculated.