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# CUSP FORMS OF WEIGHT ONE, QUARTIC RECIPROCITY AND ELLIPTIC CURVES

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## §1. Introduction

Let $m$ be a non-square positive integer. Let $K$ be the Galois extension over the rational number field $\boldsymbol{Q}$ generated by $\sqrt{-1}$ and $\sqrt[4]{m}$. Then its Galois group over $\boldsymbol{Q}$ is the dihedral group $D_4$ of order 8 and has the unique two-dimensional irreducible complex representation $\psi$. In view of the theory of Hecke-Weil-Langlands, we know that $\psi$ defines a cusp form of weight one (cf. Serre [6]). This cusp form is denoted by $\theta(\tau, K)$. The present paper consists of two parts. In the first part (§2 and §3), we shall study the number theoretic properties of $\theta(\tau, K)$ deduced from $K$. We show firstly that $\theta(\tau, K)$ has three expressions by definite or indefinite theta series. We may consider these expressions of $\theta(\tau, K)$ as the identities between cusp forms of weight one. This point of view gives a number theoretic explanation for the identities between cusp forms ([3]). Further we show that the Fourier coefficients of the cusp form $\theta(\tau, K)$ determine the decomposition law of the extension $K/\boldsymbol{Q}$ and especially the quartic residuacity of $m$. These results are obtained from that $K$ has three quadratic subfields over which $K$ is abelian. In particular, for the case $m$ is prime, we write down the above expressions of $\theta(\tau, K)$ explicitly by determining the class group corresponding to $K$ in each quadratic subfield. We deduce from this a special case of quartic reciprocity law. In this part we also establish the "higher reciprocity law" of the defining equation of $K$.

Let $E$ be the elliptic curve defined by the equation: $y^2=x^3+4mx$. Then $K$ is generated over $\boldsymbol{Q}$ by certain torsion points of $E$. The purpose of the second part is to study the property of $\theta(\tau, K)$ related to $E$ through $K$. Let $\vartheta(\tau, E)$ denote the inverse Mellin transform of the $L$-function of $E$. Then $\vartheta(\tau, E)$ is a cusp form of weight two (cf. Shimura [8]). In

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