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A REMARK ON SMITH'S RESULT ON A DIVISOR PROBLEM IN ARITHMETIC PROGRESSIONS

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§1. Introduction

Let $d_k(n)$ be the number of the factorizations of n into k positive numbers. It is known that the following asymptotic formula holds:

$$\sum_{\substack{n\leq x\\ n\equiv r(\mathrm{mod}\ q)}} d_k(n) = \varphi(q)^{-1}xP_k(\log(x)) + \Delta_k(q;r)\ ,$$

where r and q are co-prime integers with $0<r<q$, P_k is a polynomial of degree $k-1$, $\varphi(q)$ is the Euler function, and $\Delta_k(q;r)$ is the error term. (See Lavrik [3]).

In 1982, R. A. Smith [5] proved that if $(r,q)=1$, then for $x\geqq q^{\frac{1}{2}(k+1)}$,

$$\Delta_k(q;r) = F_k(0) + O(x^{(k-1)/(k+1)}(\log(2x))^{k-1}d_k(q))\ , \tag{1.1}$$

where the function $F_k(s)$ is the meromorphic continuation of the Dirichlet series

$$\sum_{n\equiv r(\mathrm{mod}\ q)} d_k(n)n^{-s}\ .$$

The proof of Smith depends essentially on Deligne's famous work concerning Weil's conjecture [1].

A remaining problem is the estimation of the term $F_k(0)$. In the "Note added in proof" of [5], Smith announced the estimate $F_k(0)\ll q^{\frac{1}{2}k}(\log(q))^k$, so the explicit upper bound of $\Delta_k(q;r)$ obtained by Smith is as follows:

$$\Delta_k(q;r) = O(q^{\frac{1}{2}k}(\log(q))^k + x^{(k-1)/(k+1)}(\log(2x))^{k-1}d_k(q))\ . \tag{1.2}$$

Furthermore, Smith conjectured that the upper bound of $F_k(0)$ can be improved to $q^{\frac{1}{2}(k-1)+\varepsilon}$ for any $\varepsilon>0$. He said, "I will return to this problem at another time." But, unfortunately, he suddenly passed away in March 1983, at forty-six years old.

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