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THE HAUSDORFF DIMENSION OF GENERAL SIERPIŃSKI CARPETS

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§1. Introduction

In this note we determine the Hausdorff dimension of a family of planar sets which are generalizations of the classical Cantor set. Given $n \geq m$ and a set R consisting of pairs of integers (i, j) with $0 \leq i < n$ and $0 \leq j < m$, define the set $\bar{R}$ by

$$\bar{R} = \left\{ \left(\sum_1^\infty \frac{x_k}{n^k}, \sum_1^\infty \frac{y_k}{m^k} \right) : (x_k, y_k) \in R \,\forall k \right\} .$$

We refer to $\bar{R}$ as a general *Sierpiński carpet*, after Mandelbrot [4], since Sierpiński's universal curve is a special case of this construction [6].

It is clear that $\bar{R} = \bigcup_1^r f_i(\bar{R})$, where $r = |R|$ and the f_i are affine maps contracting $\bar{R}$ by a factor of n horizontally and m vertically. When $n = m$ these maps are actually similarity transformations, and a well-known argument shows the dimension of $\bar{R}$ is $\log r / \log n$ (following e.g. Beardon [1]). If $n > m$, however, a different approach is required, essentially because squares are stretched into narrow rectangles under iteration of the maps f_i.

Our method relies on elementary probability theory to address the general case, and we obtain the following result.

THEOREM. *The Hausdorff dimension of $\bar{R}$ is given by*

$$\dim \bar{R} = \log_m \left(\sum_{j=0}^{m-1} t_j^{(\log_n m)} \right)$$

where t_j is the number of i such that $(i, j) \in R$.

This settles a question of Hironaka's [2] concerning the dimension of a certain continuous plane curve whose self-similarities entail the

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