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ON McKAY'S CONJECTURE

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Let $\eta(z)$ be Dedekind's η-function. For any set of integer $g=(k_1,\cdots,k_s)$, $k_1\geq k_2\geq\cdots\geq k_s\geq 1$, put $\eta_g(z)=\prod_{i=1}^s\eta(k_iz)$. In this paper, we shall prove McKay's conjecture which gives some combinatorial conditions about k_i on which $\eta_g(z)$ is a primitive cusp form. As to McKay's conjecture, we refer [5].

To state our result precisely, we introduce some notation. For every positive integer N, put

$$\Gamma_0(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,Z)\,|\,c\equiv 0 \bmod N\right\}.$$

Let k be a positive integer and let ε be a Dirichlet character mod N such that $\varepsilon(-1)=(-1)^k$. We denote by $S_k(N,\varepsilon)$ (resp. $S_k^0(N,\varepsilon)$) the space of the cusp forms (resp. new forms) of type (k,ε) on $\Gamma_0(N)$. We call $f(z)=\sum_{n=1}^\infty a_ne(nz)$ in $S_k^0(N,\varepsilon)$ primitive cusp form if it is a common eigenfunction of all the Hecke operators and $a_1=1$ where $e(z)=e^{2\pi iz}$. Then it is well-known that $S_k^0(N,\varepsilon)$ has a basis whose elements are all primitive cusp forms.

McKay conjectured

THEOREM. *Let $\eta_g(z)=\prod_{i=1}^s\eta(k_iz)$ be as above. The following statements* (a) *and* (b) *are equivalent.*

(a) *$\eta_g(z)$ is a primitive cusp form.*

(b) *$g=(k_1,\cdots,k_s)$ satisfies the conditions* (1)$\sim$(4);

(1) *k_1 is divisible by k_i for all $1\leq i\leq s$.*

(2) *Put $N=k_1k_s$, then $N/k_i=k_{s+1-i}$ for all $1\leq i\leq s$.*

(3) $\sum_{i=1}^s k_i=24$.

(4) *s is even.*

In these cases, $\eta_g(z)$ is a primitive cusp form in $S_{s/2}^0(k_1k_s,\varepsilon)$ for some Dirichlet character ε mod k_1k_s.