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INTEGRABLE DERIVATIONS

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Dedicated to Prof. Yoshikazu Nakai on his Sixtieth Birthday

Introduction

Let A be a commutative ring and D be a derivation of A into itself. If there exists a homomorphism $E: A \to A[[t]]$ such that

$$E(a) \equiv a + tD(a) \qquad \bmod t^2$$

then we say that D is integrable. Integrable derivations have many good properties. In fact, most of unpleasant phenomena of derivations in characteristic p disappear if we consider integrable derivations only.

In § 1 we state definitions and basic properties of differentiations, and we give some examples of non-integrable derivations.

§ 2 is devoted to theorems which are essentially due to Seidenberg ([18], [19], [20]). These theorems show that integrable derivations behave as they should, and provides us with necessary conditions for integrability.

Then in § 3 and § 4 we prove some sufficient conditions. In § 3 we consider smooth or formally smooth algebras, using André's homology theory. In § 4, by an elementary argument we prove a criterion of integrability, which shows that there are plenty of integrable derivations (in the case of an integral domain finitely generated over a perfect field).

§ 1. Definitions and examples

In this article all rings are assumed to be commutative with a unit element. Local rings are assumed to be noetherian.

Let A be a ring. The set of all derivations of A into itself is an A-module and is denoted by $\mathrm{Der}(A)$. If k is a subring of A, the submodule of $\mathrm{Der}(A)$ consisting of those derivations which vanish on k is denoted by $\mathrm{Der}_k(A)$.

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